

Geometric Flows of Diffeomorphisms

Anthony Carapetis

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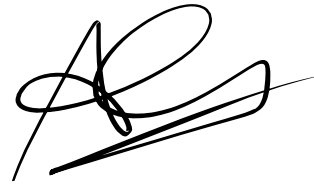
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Declaration

The work in this thesis is my own except where otherwise stated.

This thesis contains approximately 30,000 words.

A handwritten signature in black ink, consisting of a stylized 'A' followed by a series of loops and a long horizontal stroke.

Anthony Carapetis

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Finally, I'd like to thank my parents and my sister for being the best family I could hope for. Yes, I finally wrote that "paper".

Abstract

The idea of this thesis is to apply the methodology of geometric heat flows to the study of spaces of diffeomorphisms. We start by describing the general form that a geometrically natural flow must take and the implications this has for the evolution equations of associated geometric quantities. We discuss the difficulties involved in finding appropriate flows for the general case, and quickly restrict ourselves to the case of surfaces. In particular the main result is a global existence, regularity and convergence result for the flow $\partial_t u = \left(|Du|^2 + 2|\det Du|\right)^{-1} \Delta u$ of maps u between flat surfaces, producing a strong deformation retract of the space of diffeomorphisms onto a finite-dimensional submanifold. Partial extensions of this result are then presented in several directions. For general Riemannian surfaces we obtain a full local regularity estimate under the hypothesis of bounds above and below on the singular values of the first derivative. We achieve these gradient bounds in the flat case using a tensor maximum principle, but in general the terms contributed by curvature are not easy to control. We also study an initial-boundary-value problem for which we can attain the necessary gradient bounds using barriers, but the delicate nature of the higher regularity estimate is not well-adapted for obtaining uniform estimates up to the boundary. To conclude, we show how appropriate use of the maximum principle can provide a proof of well-posedness in the smooth category under the assumption of estimates for all derivatives.

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Chapter 1

Introduction

Geometric heat flows are now a standard tool in differential geometry. Whenever the problem of the existence of some “nice” object can be posed as an elliptic PDE, it is natural to view this as the steady-state version (or long-time limit) of a corresponding parabolic problem. In many (but certainly far from all) cases these nice objects are the ones that minimize some objective functional, and the parabolic equation is taken to be an appropriate gradient flow of this functional. The heat flow method has been famously applied to produce harmonic maps and constant-curvature metrics via the harmonic map heat flow (starting with [ES64]; the L^2 gradient flow of the Dirichlet energy) and Ricci flow (starting with [Ham82]; not a gradient flow¹) respectively. The general program is to establish existence of solutions to the flow (typically requiring both geometric and analytic methods), and then to determine their long-time behaviour. In the simplest case we get long-time convergence to a nice object; so the solution of the flow constitutes a path in the space of all objects joining the initial datum to something nice.

In addition to this “pointwise” application of showing that any given object can be deformed to a nice one (that is, the existence of steady-state solutions/minimizers in homotopy classes), these flows also provide topological information about how the locus of nice objects sit inside the space of all objects: by flowing every object at once, the flow produces a deformation retract of the space on to the locus (so long as we have good enough estimates to prove that the solution operator is continuous). In particular, we find that the locus of nice objects is homotopy equivalent to the space of all objects. Since we often have a very concrete understanding of this locus of the nice objects (including the homotopy type), we can draw purely (differential) topological conclusions about the full space.

The idea behind this thesis is to apply these techniques to the study of diffeomorphisms, the natural symmetries of smooth manifolds. Here is one classic topological result motivating such a program:

¹The Ricci flow is not a gradient flow for the metric alone, and was usefully studied for over a decade without needing any such structure. The famous work of Perelman [Per02] eventually showed that a slight extension of the Ricci flow could be viewed as a gradient flow for the metric along with a scalar function known as the *dilaton field*.

Theorem (Smale’s Theorem on diffeomorphisms of S^2 [Sma59]). *The space $\text{Diff}^+(S^2)$ of orientation-preserving diffeomorphisms of the two-dimensional sphere has a strong deformation retract on $SO(3)$, the group of orientation-preserving isometries of the sphere.*

This was originally proved in two steps: first, notice that $\text{Diff}^+(S^2) \simeq SO(3) \times \text{Diff}(\bar{D}^2, \partial D^2)$. Here \bar{D}^2 is the closed disc and $\text{Diff}(\bar{D}^2, \partial D^2)$ denotes the set of diffeomorphisms of \bar{D}^2 that restrict to the identity on the circle ∂D^2 . This isomorphism has a simple geometric description: by composing any diffeomorphism of S^2 with the right rotation we can ensure that it fixes the north pole, and that its differential there is the identity. Removing the north pole and “unwrapping” the rest of the sphere to an open disc so that the pole corresponds to the boundary, these two properties mean that the corresponding diffeomorphism of the open disc will extend to an element of $\text{Diff}(\bar{D}^2, \partial D^2)$. The problem is thus reduced to the second step: show that $\text{Diff}(\bar{D}^2, \partial D^2)$ is contractible². Smale’s original proof of this (actually, the equivalent statement for the closed square) used the flows of several vector fields along with some very particular properties of the behaviour of first-order ODEs in two dimensions.

To give a proof by flow, the most obvious idea is to look for a flow that deforms a diffeomorphism of the sphere to a rotation. Alternatively, we could replace just the second part of Smale’s proof by finding a flow that deforms an element of $\text{Diff}(D^2, \partial D^2)$ to the identity. We will explore both of these paths, though unfortunately they both hit roadblocks: the first when controlling the effects of curvature, and the second due to the effects of the boundary on our regularity theory.

A more ambitious idea would be to do something similar in higher dimensions. The natural generalization $\text{Diff}^+(S^3) \sim SO(4)$ is most commonly known as the Smale conjecture, and was proven by Hatcher [Hat83] by a very involved argument. Finding a heat flow proof of this would be great, but this is a long way off: the results of this thesis say essentially nothing about dimensions greater than two. The corresponding problem is open in dimension 4 and known to be false in some higher dimensions [Hat12].

There are also more practical reasons to flow diffeomorphisms of surfaces: in medical imaging and computer graphics, the problem of *surface matching* is to find a diffeomorphism (or some discrete approximation thereof) between two surfaces that is in some sense close to an isometry; i.e. minimizes some distortion measurement. One emerging approach [SS15, AL15] is to apply optimization methods to functionals such as the “symmetric³ Dirichlet energy” $J(u) = \int (|Du|^2 + |Du|^{-2}) dA$, so it would be reassuring to have some results on the well-posedness of these variational problems and their associated gradient flows. There are some similar results known for maps between plane domains [DGM98, ESG05], but for surfaces it seems that the theory is just not there.

²This reduction really works in any dimension: the obvious generalization of this argument shows $\text{Diff}(S^n) \simeq SO(n+1) \times \text{Diff}(\bar{D}^n, \partial D^n)$. Thus the second step is really the non-trivial part of this theorem.

³This energy is actually not symmetric under inversion of the map u : perhaps $\int (|Du|^2 dA + |Du|^{-2} u^* dA)$ is more deserving of this name.

Now that we have some motivation, we will briefly discuss the kind of flows we will be considering. Since a diffeomorphism $u_0 \in \text{Diff}(M)$ is a smooth map $M \rightarrow M$, the well-studied harmonic map heat flow

$$\partial_t u = \Delta u$$

defines a family $u : M \times [0, T) \rightarrow M$ of maps with initial condition u_0 ; but there is no guarantee that later time slices $u|_{M \times \{t\}}$ will also be diffeomorphisms. We will see later (in Example 3.2.2) that the loss of the diffeomorphism property can in fact occur, and thus this flow is not well-suited to studying diffeomorphisms. This is a great shame - the natural form of the harmonic map heat flow makes it exceptionally well-behaved, with many estimates available essentially for free.

In certain special cases other flows are known to work. One option is to work with an extrinsic flow for the graph of u in $M \times M$, which was the approach taken by Wang in [Wan01]: he showed that if M is a compact surface of constant curvature and u is locally area-preserving (i.e. pulls back the area form) then the mean curvature flow of $\text{graph}(u)$ remains the graph of an area-preserving diffeomorphism, exists for all time, and converges to a so-called *minimal map*. In particular, when M is the round 2-sphere, the limit is an isometry. Since the diffeomorphism group of any compact manifold retracts on to the area-preserving diffeomorphisms [Mos65], this works as a nice proof of Smale's theorem; but it would be even more pleasing to find a flow that does the whole job at once.

In the present work we take the route of modifying the harmonic map heat flow, i.e. working directly with the map u . In order to inherit at least some of the nice properties of the harmonic map heat flow, we want to stay as close to it as is possible. Thus we study the family of quasilinear flows given by equations of the form

$$\partial_t u^\alpha = a^{ij} (Du) \nabla_i \nabla_j u^\alpha.$$

(We use the summation convention for repeated indices.) The idea is that careful choice of the dependence of the diffusion coefficients a^{ij} on the derivative will allow us to get lower bounds on the derivative, and in particular preserve the fact that u is a diffeomorphism - this is the topic of Chapter 4. The fact that this system is in some sense diagonal (as opposed to the more general equation $\partial_t u^\alpha = a^{ij\alpha}_\beta \nabla_i \nabla_j u^\beta$ we could write down) means that the corresponding evolution equation for Du will have the same property, which allows us to apply the maximum principle to determine which coefficients we should choose. (As a downside, this means we won't be able to say much about the gradient flows we discussed above.)

Before we can get to this, we need to discuss the natural assumptions on a^{ij} imposed by isometry invariance and the resulting evolution equations for the natural geometric scalars σ_i , which are the topic of Chapter 3. The failure of the singular value decomposition to be everywhere smooth throws a bit of a spanner in the works here; but we will show that the

isometry invariance smooths out all the issues.

Unfortunately, making this move away from harmonic map heat flow comes with a downside - we lose a great deal of the regularity theory, and in particular no longer have Bernstein-type estimates available for the higher derivatives. Thus the regularity theory for our flows takes on more of the flavour of nonlinear PDE systems: we need to prove our own estimates in order to rule out finite-time blowup of the flow, which is the topic of Chapter 5. As is typical for quasilinear equations, we just need to obtain Hölder estimates for the coefficients $a^{ij}(Du)$, after which a Schauder bootstrap argument provides full regularity.

We will see that when the initial map $\mathbb{R}^n \rightarrow \mathbb{R}^n$ is *exact* (i.e. the gradient of some scalar function), the dynamics of a special class of flows are determined by the evolution of a single scalar function obeying a nonlinear parabolic PDE: If $u : \mathbb{R}^n \times [0, T) \rightarrow \mathbb{R}^n$ is the (spatial) gradient $\phi : \mathbb{R}^n \times [0, T) \rightarrow \mathbb{R}$ solving $\partial_t \phi = f(D^2 \phi)$, then u solves

$$\partial_t u^\alpha = \frac{\partial f}{\partial \phi_{ij}}(Du) \nabla_i \nabla_j u^\alpha.$$

With the right assumptions on f , we can apply the existing theory for scalar PDE to obtain Hölder estimates for u . By thinking of general data in terms of some measurement of their deviation from exactness, we will find that the flow

$$\partial_t u = \frac{\Delta u}{(\sigma_1 + \sigma_2)^2} = \frac{\Delta u}{|Du|^2 + 2 \det Du} \quad (1.0.1)$$

(corresponding to $\partial_t \phi = -1/\Delta \phi$) makes this deviation measurement obey a scalar PDE, and use the resulting information about this deviation in order to establish the desired estimate for the coefficients. We also show that despite the originating case of exact diffeomorphisms making sense only in \mathbb{R}^n , this estimate can be adapted to work on an arbitrary Riemannian surface.

The problem of satisfying both of these estimates (diffeomorphism-preserving and higher regularity) is quite difficult, and in this thesis we will only achieve both for the case when $M \simeq \mathbb{T}^2$ is a flat 2-dimensional torus and the flow is (1.0.1); i.e. $a^{ij} = (|Du|^2 + 2 \det Du)^{-1}$. Here σ_1, σ_2 are the singular values of the derivative Du , which can be thought of as the minimum and maximum local length distortions of the map u . The flatness is necessary to get bounds on the singular values, while the particular choice of flow is required to make the Hölder estimate work. In this case (actually a slight generalization that allows different geometries on the domain and target) we will prove a complete existence and convergence theorem:

Theorem 1.0.1. *For any flat 2-dimensional tori M, N and any $u_0 \in \text{Diff}(M, N)$, there is a smooth solution $u : [0, \infty) \rightarrow \text{Diff}(M, N)$ of the flow (1.0.1), and $u(t)$ converges smoothly to a harmonic diffeomorphism as $t \rightarrow \infty$. Moreover, the solution map sending (u_0, t) to $u(t)$ is continuous in the C^∞ topology, so the flow constitutes a strong deformation retract of $\text{Diff}(M, N)$ on to the finite-dimensional space of harmonic diffeomorphisms.*

See Figure 1.0.1 for an illustration of this flow when $M = N$ is a square torus.

Along the way to completing the proof of this theorem, we will highlight the difficulties involved in generalizing it to more interesting settings, and provide partial results in several of these directions. The existence and convergence parts of Theorem 1.0.1 previously appeared as Theorem 1 in [AC16], which contained just the essential estimates required to prove this particular result, which will appear here in sections 4.2, 5.3 and 6.2. This thesis can thus be seen as filling two kinds of gaps in [AC16]:

- Chapters 3–5 expose the considerations that led us to arrive at the particular choice of flow equation (1.0.1), and provide some original results unnecessary for the proof of Theorem 1.0.1 but that could be useful for future work.
- Chapters 2 and 6 provide some of the required background and standard results in geometric analysis that are familiar to experts in the field.

All of the original research described in this thesis was in collaboration with Ben Andrews.

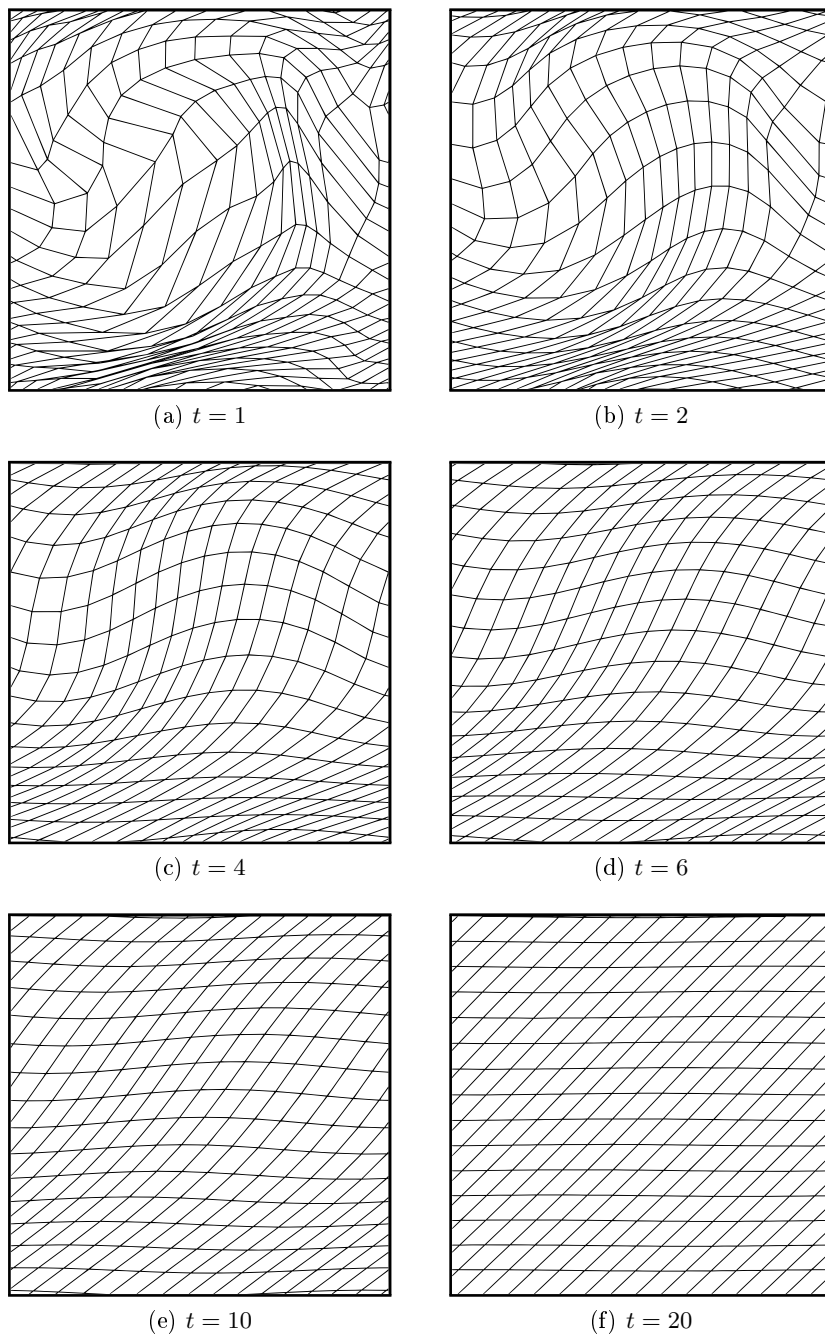


Figure 1.0.1: Approximate solution of (1.0.1) on a square torus, visualized by sketching the image under u of a regular square grid at a sequence of times t . Note the similarities with the classical heat equation: we observe both rapid smoothing of high-frequency distortions and long-time convergence to a “straight” limit (here some composition of a translation with the affine diffeomorphism $(x, y) \mapsto (x - y, y)$).

You can play with this flow yourself: visit http://a.carapetis.com/diff_flow/ for an interactive version.

Chapter 2

Background: Geometry and PDE

“Differential geometry is the study of properties that are invariant under change of notation.”

Ancient Cliché

We will now very briefly develop the standard geometric and analytic machinery that we will need to study flows of maps. This chapter serves mostly as a reference to our notation and conventions - differential geometry in particular is famous for being expressed in many different forms, so it's best we make sure that we're on the same page now. None of the material in this chapter is original. The experienced reader should skip directly to Chapter 3 and treat this chapter as a reference if necessary. If these subjects are unfamiliar, the reader is advised to learn them from a source consisting of more than just this dry list of definitions. I have recommended some of my favourite such sources at the beginning of each section, but there are of course many more excellent texts available on these topics than I could list.

2.1 Geometric Structures on Vector Bundles

We will use the theory of affine connections on general vector bundles, since to work with derivatives of maps between manifolds we will need to extend our geometric structure to products and pullbacks of the tangent bundle. We will occasionally need to work with smooth (non-vector) fibre bundles that arise as submanifolds of vector bundles. For the reader unfamiliar with this material, I recommend [Lee03, Lee97] as an introduction; while for a comprehensive reference one can use [KN63, KN69]. The classical results of Riemannian geometry (such as can be found in [O'N83] or [CF92]) are assumed.

We assume the standard terminology of differential topology, and that manifolds are Hausdorff, second-countable and connected unless specified otherwise. We will sometimes use the notation M^m to quickly note that M has dimension m - if we mean to take a product, we will write it explicitly as $M \times M$ rather than M^2 .

A chart is a diffeomorphism¹ $\phi : U \rightarrow \phi(U)$ where $U \subset M$ and $\phi(U) \subset \mathbb{R}^m$ are open; that is, our convention is that charts are coordinates and inverses of charts are parametrizations. An atlas for M is a collection of charts $\{\phi_i\}_{i \in I}$ such that the corresponding domains U_i cover M . A finite atlas is just an atlas consisting of a finite number of charts, i.e. with $|I| < \infty$.

For $k, l \in \mathbb{N}$ and E a vector space or bundle, we write

$$\otimes^k E = \underbrace{E \otimes \dots \otimes E}_k$$

for the k -fold tensor product and $T_l^k E = (\otimes^k E) \otimes (\otimes^l E^*)$ for the space/bundle of (k, l) -tensors over E . The empty tensor product of spaces is \mathbb{R} and likewise that of bundles over M is the trivial line bundle $\mathbb{R}_M = M \times \mathbb{R}$. The k -fold symmetric and antisymmetric tensor products of E will be denoted by $\text{Sym}^k E$ and $\Lambda^k E$ respectively.

The ring of smooth real-valued functions on M is denoted by $C^\infty(M)$. We will denote the $C^\infty(M)$ -module of smooth sections of E by $\Gamma(E)$, while $s \in \Gamma_{\text{loc}}(E)$ means that s is a section of $E|_U$ for some open domain $U \subset M$. If we need to work with the space of sections of a given C^k regularity we will instead write $C^k(M, E)$ or $C_{\text{loc}}^k(M, E)$.

We will often use abstract index notation, so that e.g. $\nabla_i \nabla_j f$ denotes the second covariant derivative $\nabla^2 f = (\nabla_i (\nabla_j f) - \nabla_{\nabla_i \partial_j} f) dx^i \otimes dx^j$. (In Chapter 3 we will extend this notation to $\nabla_i \nabla_j u^\alpha$ for maps between manifolds.) We use $T_{[ij]} = \frac{1}{2}(T_{ij} - T_{ji})$ and $T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji})$ to denote skew-symmetrization and symmetrization respectively.

Definition 2.1.1. A *(fibre) metric* on a vector bundle $E \rightarrow M$ is a section $h \in \Gamma(E^* \otimes E^*)$ that is *positive-definite*; i.e. such that $h(\xi, \xi) > 0$ for all non-zero $\xi \in E$.

Definition 2.1.2. A *(Riemannian) metric* on a smooth manifold M is a fibre metric on TM . A *Riemannian manifold* is a pair (M, g) where M is a smooth manifold and g is a metric on M .

We will often leave the metric implicit, with the symbol M alone referring to the manifold equipped with its metric.

Definition 2.1.3. A *connection* on a vector bundle $E \rightarrow M$ is an additive map $\nabla : \Gamma(E) \rightarrow \Gamma(TM^* \otimes E)$ satisfying the Leibniz rule

$$\nabla(f\xi) = df \otimes \xi + f \nabla \xi$$

for all $f \in C^\infty(M)$ and $\xi \in \Gamma(E)$. We will write $\nabla_X \xi = \nabla \xi(X) = X^i \nabla_i \xi^A e_A$.

¹Since we're using the notion of a diffeomorphism of smooth manifolds already, the "definitions" of charts and atlases given here are post-hoc, and unsuitable for developing smooth structures from first principles. The goal here is simply to make our conventions clear.

Proposition 2.1.4. *Given a vector bundle $E \rightarrow M$ equipped with a connection ∇ , there is a unique extension of ∇ to the tensor algebra*

$$\mathcal{T}(E) = \bigoplus_{k=0}^{\infty} \bigoplus_{l=0}^{\infty} T_l^k E$$

respecting the grading (i.e. $\nabla_X (T_l^k E) \subset T_l^k E$) that commutes with contractions and satisfies the Leibniz rule $\nabla(\xi \otimes \eta) = \nabla\xi \otimes \eta + \xi \otimes \nabla\eta$.

Proof. The last two laws imply that for any $\xi \in \Gamma(E), \theta \in \Gamma(E^*)$ we have $d(\theta(\xi)) = d(C(\theta \otimes \xi)) = C(\nabla(\theta \otimes \xi)) = \nabla\theta(\xi) + \theta(\nabla\xi)$, which uniquely determines ∇ on E^* . The Leibniz rule then uniquely determines ∇ on all tensor products of E and E^* , and additivity extends this to the whole tensor algebra. \square

We will often work with this extension implicitly, as in the next definition where we use the connection induced on $E^* \otimes E^*$ by that on E .

Definition 2.1.5. A connection ∇ and a fibre metric h on $E \rightarrow M$ are *compatible* (or ∇ is a *metric connection* for h) if $\nabla h = 0$; i.e. if $\nabla_X h(\xi, \eta) = h(\nabla_X \xi, \eta) + h(\xi, \nabla_X \eta)$ for all $X \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(E)$.

Definition 2.1.6. The *torsion* of a connection ∇ on TM is the tensor

$$\tau^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Theorem 2.1.7 (Miracle of Riemannian Geometry). *On a Riemannian manifold (M, g) , there is a unique connection ∇ on TM (the Riemannian connection or Levi-Civita connection) that is torsion-free ($\tau^\nabla = 0$) and compatible with the metric ($\nabla g = 0$).*

Proposition 2.1.8. *Given two bundles $E^{(1)} \rightarrow M, E^{(2)} \rightarrow M$ equipped with connections $\nabla^{(1)}$ and $\nabla^{(2)}$ respectively, there is a unique natural connection $\nabla = \nabla^{(1)} \otimes \nabla^{(2)}$ on $E^{(1)} \otimes E^{(2)}$ satisfying the Leibniz rule*

$$\nabla_X (\xi \otimes \eta) = \nabla_X^{(1)} \xi \otimes \eta + \xi \otimes \nabla_X^{(2)} \eta.$$

If $\nabla^{(i)}$ are metric connections for $h^{(i)}$, then $\nabla^{(1)} \otimes \nabla^{(2)}$ is compatible with $h^{(1)} \otimes h^{(2)}$. Similarly we obtain a unique sum connection $\nabla = \nabla^{(1)} \oplus \nabla^{(2)}$ on $E^{(1)} \oplus E^{(2)}$ by

$$\nabla_X (\xi + \eta) = \nabla_X^{(1)} \xi + \nabla_X^{(2)} \eta$$

which is compatible with $h^{(1)} + h^{(2)}$.

Proposition 2.1.9. *Given a bundle $E \rightarrow N$ equipped with a connection ∇ and a smooth map $u : M \rightarrow N$, there is a unique connection ${}^u\nabla$ (the pullback connection) on uE satisfying ${}^u\nabla_X (\xi_u) = (\nabla_{u_* X} \xi)_u$ for all $\xi \in \Gamma(N), X \in \Gamma(TM)$. (Here ξ_u denotes the*

restriction of ξ to u ; i.e. the composition $\xi \circ u$, which is naturally a section of u^*E .) If ∇ is compatible with a metric h , then ${}^u\nabla$ is compatible with the metric h induces on u^*E .

Unless we explicitly state otherwise, products/duals/pullbacks of bundles with geometric structure are implicitly equipped with the corresponding natural metrics and connections as defined above. We use $\langle \text{angle, brackets} \rangle$ to denote the natural metric on any bundle, and similarly ∇ for the natural connection. For any bundle E equipped with a fibre metric h we will sometimes use the *musical isomorphisms*

$$\begin{aligned} \flat & : E \rightarrow E^* \\ \sharp = \flat^{-1} & : E^* \rightarrow E \end{aligned}$$

defined by $\xi^\flat(\eta) = h(\xi, \eta)$, which correspond to raising/lowering indices with the metric.

Definition 2.1.10. The *curvature* of a connection ∇ on $E \rightarrow M$ is the section

$$R^\nabla \in \Gamma(TM^* \otimes TM^* \otimes E^* \otimes E)$$

defined by

$$R^\nabla(X, Y)\xi = \nabla^2\xi(X, Y) - \nabla^2\xi(Y, X) = \nabla_X(\nabla_Y\xi) - \nabla_Y(\nabla_X\xi) - \nabla_{[X, Y]}\xi.$$

When ∇ is the Levi-Civita connection of a Riemannian manifold M , then $R^M = R^\nabla$ is the *Riemann curvature* of M .

Note that this is antisymmetric in X, Y , so we can also consider it as the *curvature form* $R^\nabla \in \Omega^2(M; \text{End}(E))$. In coordinates x^i on M and a frame e_A for E it is determined by the formula

$$2\nabla_{[i}\nabla_{j]}e_A = R_{ij}{}^B{}_A e_B. \quad (2.1.1)$$

We will often be interested in the curvature of various derived bundles:

Proposition 2.1.11. *Let E, F be vector bundles over N equipped with a connections and let $u : M \rightarrow N$ be a smooth map. Then the curvature of the natural connection on E^* is given by*

$$R(X, Y, \theta)\xi = -\theta R^E(X, Y, \xi),$$

on $E \otimes F$ by

$$R(X, Y, \xi \otimes \eta) = R^E(X, Y, \xi) \otimes \eta + \xi \otimes R^F(X, Y, \eta)$$

and on u^*E by

$$R(X, Y, \xi_u) = (R^E(u_*X, u_*Y, \xi))_u.$$

Proof. Since ∇ commutes with contractions and satisfies the Leibniz rule, for any $\theta \in \Gamma(E^*)$, $\xi \in \Gamma(E)$ we have

$$0 = \nabla_{[i}\nabla_{j]}(\theta(\xi)) = (\nabla_{[i}\nabla_{j]}\theta)(\xi) + \theta(\nabla_{[i}\nabla_{j]}\xi)$$

because second derivatives of the scalar $\theta(X)$ commute. From the definition of curvature this implies $R_{ijA}^{E*B} = -R_{ij}^B{}_A$. An almost identical calculation (just without the contraction) shows the curvature of $\theta \otimes \eta$ is

$$\nabla_{[i}\nabla_{j]}\theta \otimes \eta + \theta \otimes \nabla_{[i}\nabla_{j]}\eta.$$

Finally, compute

$$\begin{aligned} R^{u\nabla}(\partial_i, \partial_j, \xi_u) &= {}^u\nabla_{\partial_i} \left(\nabla_{u_*\partial_j} \xi \right)_u = \left(u_{[i}^\alpha \nabla_\alpha \left(u_{j]}^\beta \nabla_\beta \xi \right) \right)_u \\ &= \left(u_{[i}^\alpha u_{j]}^\beta \nabla_\alpha \nabla_\beta \xi + u_{[ij]}^\beta \nabla_\beta \xi \right)_u = \left(u_i^\alpha u_j^\beta \nabla_{[\alpha} \nabla_{\beta]} \xi \right)_u = u_i^\alpha u_j^\beta (R^\nabla(\partial_\alpha, \partial_\beta, \xi))_u \\ &= (R^\nabla(u_*\partial_i, u_*\partial_j, \xi))_u \end{aligned}$$

as desired. \square

By taking the curvature of a Riemannian connection we obtain the usual Riemannian curvature tensor $R_{ij}{}^k{}_l \in \Gamma(T_3^1 TM)$ and its completely covariant version $R_{ijkl} = R_{ij}{}^a{}_l g_{ak} \in \Gamma(T_4^0 TM)$, with its well-known symmetries $R_{ijkl} = -R_{jikl} = R_{klij}$.

Contracting the Riemannian curvature gives us the Ricci and scalar curvatures:

Definition 2.1.12. The *Ricci curvature* of a Riemannian manifold (M, g) is the $(0,2)$ -tensor $\text{Rc} = R_{ij} = R_{ki}{}^k{}_l$. The *scalar curvature* is the function $\text{Scal} = \text{tr}_g \text{Rc} = g^{ij} R_{ij}$.

The symmetries of the Riemannian curvature imply that it can be considered as a symmetric bilinear form on $\Lambda^2 TM$ determined by $R(u \wedge v, w \wedge x) = R(u, v, w, x)$. Remembering that the Grassmannian $G(E, k)$ is the bundle whose fibre over p is the set of k -dimensional subspaces of E_p , we can define the curvature of a plane by representing it as a bivector via the Plücker embedding:

Definition 2.1.13. The *sectional curvature* of a subspace $\Pi \in G(TM, 2)$ is defined by

$$K(\Pi) = K(u \wedge v) = R(u \wedge v, u \wedge v)$$

where $\{u, v\}$ is any orthonormal basis for Π .

We say M has *constant curvature* if $K : G(TM, 2) \rightarrow \mathbb{R}$ is constant. In dimensions $n \geq 3$ this is equivalent to K being constant on each tangent space (i.e. K factoring as $G(TM, 2) \xrightarrow{\pi} M \rightarrow \mathbb{R}$), a fact sometimes known as *Schur's Lemma*. (See e.g. [Jos02, Theorem 4.3.2] for a proof.) If M is a surface, this is not true; but there is only a single subspace $\Pi = T_p M$ for each point $p \in M$; so the complete content of the Riemannian curvature is a single scalar function:

Definition 2.1.14. The *Gaussian curvature* of a surface M is the function $\kappa : M \rightarrow \mathbb{R}$ defined by $\kappa(p) = K(T_p M)$.

Proposition 2.1.15. *If M is a surface, the curvatures are given by $R_{ijkl} = \kappa(g_{ik}g_{jl} - g_{il}g_{jk})$, $\text{Rc} = \kappa g$ and $\text{Scal} = 2\kappa$.*

Proof. If we work in an orthonormal frame (so that $g = \delta$ and all indices can be lowered/raised for free), we see that $\kappa = K(e_1 \wedge e_2) = R_{1212}$. By the symmetries of R , its only non-zero components are $R_{1212} = R_{2121} = -R_{1221} = -R_{2112} = \kappa$. Replacing g with δ we see that these are exactly the non-zero components of our desired expression. Contracting this with g^{jl} gives $R_{ik} = \kappa(\delta_{ik}\delta_{jl}\delta_{jl} - \delta_{il}\delta_{jk}\delta_{jl}) = \kappa(2\delta_{ik} - \delta_{ik}) = \kappa g_{ik}$, and contracting once more gives $\text{Scal} = g^{ik}\kappa g_{ik} = 2\kappa$. \square

2.2 Jets

Since we will be dealing with geometrically defined nonlinear equations for maps between manifolds, we need a way of formulating these equations without dependence on coordinates. This is provided by the theory of *jets*, which are the coordinate-free generalization of Taylor polynomials. A thorough explanation of the following definitions and notations can be found in [KMS93]. Let M, N be smooth manifolds and $r \geq 0$ an integer.

Definition 2.2.1. Two curves $\gamma, \beta \in C^\infty((-\epsilon, \epsilon), M)$ have r^{th} order contact at zero (written $\gamma \sim^r \beta$) if $\varphi \circ \gamma - \varphi \circ \beta$ vanishes to r^{th} order at zero for every smooth φ ; i.e. if

$$\left(\frac{d}{dt} \right)^k (\varphi \circ \gamma(t)) - \varphi \circ \beta(t) \Big|_{t=0} = 0$$

for all $k \in \{0 \dots r\}$ and all $\varphi \in C^\infty(M)$.

This definition is quite natural - note that it is equivalent to coordinate representatives of γ, β having the same r^{th} Maclaurin polynomial. We can use this idea of contact of curves to define the contact of more general maps:

Definition 2.2.2. Two maps $f, g : M \rightarrow N$ have r^{th} order contact at $x \in M$ (written $f \sim_x^r g$) if $f \circ \gamma \sim^r g \circ \gamma$ for every smooth curve γ in M with $\gamma(0) = x$.

This is equivalent to the coordinate representatives of f, g having the same r^{th} Taylor polynomial at x . Since we want to talk about these “polynomials” themselves, we form the quotient:

Definition 2.2.3. The r -jet of $f : M \rightarrow N$ at $x \in M$ is the equivalence class

$$j_x^r f = \{g \in C^\infty(M, N) : f \sim_x^r g\},$$

which is an element of the quotient

$$J_x^r(M, N) = C^\infty(M, N) / \sim_x^r.$$

If $f(x) = y$, we say $j_x^r f$ has *source* x and *target* y . The subset of $J_x^r(M, N)$ consisting of jets with target y is denoted by $J_x^r(M, N)_y$, and the set of all r -jets from $M \rightarrow N$ with target y is denoted by $J^r(M, N)_y$. The set of all r -jets from $M \rightarrow N$ is the *jet bundle*

$$J^r(M, N) = \bigsqcup_{x \in M} \bigsqcup_{y \in N} J_x^r(M, N)_y$$

which has a natural fibre bundle structure $J^r(M, N) \rightarrow M \times N$ given by $j_x^r f \mapsto (x, f(x))$. By restricting this to preimages of slices $\{x\} \times N$ and $M \times \{y\}$ we also get vector bundles $J_x^r(M, N) \rightarrow N$ and $J^r(M, N)_y \rightarrow M$, and by composing with the natural projections we get fibrations $J^r(M, N) \rightarrow M$, $J^r(M, N) \rightarrow N$.

Definition 2.2.4. The *r-jet prolongation* (or *r-graph*) of $f : M \rightarrow N$ is the section $j^r f$ of $J^r(M, N) \rightarrow M$ defined by $j^r f(x) = j_x^r f$.

Prolongation provides a natural language for talking about differential operators acting on general mappings: for example, we can formulate a k^{th} -order differential equation for maps $u : M \rightarrow N$ as a submanifold $\Sigma \subset J^k(M, N)$, so that u is a solution if and only if $j^k u$ has image lying in Σ . For a more detailed description of this point of view, see the comprehensive text [BCG⁺91].

For our purposes we are most concerned with 1-jets, which are what we need to define second-order quasilinear differential operators on manifolds: knowing $j_x^1 f$ is equivalent to knowing exactly $f(x)$ and $Df(x)$, so $J^1(M, N)$ will be the natural domain for a in an operator of the familiar form

$$a^{ij}(x, u(x), Du(x)) \nabla_i \nabla_j u(x)$$

acting on $u : M \rightarrow N$. This also means that $J_x^1(M, N)_y \simeq T_x M^* \otimes T_y N$, and we can in fact construct the first jet bundle from the tangent bundles by gluing together these tensor products: we have $J^1(M, N) = (TM^* \times N) \otimes (M \times TN)$ where the two factors are made into vector bundles over $M \times N$ by the projection maps $\pi_{TM^*} \times \text{id}_N$ and $\text{id}_M \times \pi_{TN}$. In the language of pullback bundles this can be neatly expressed as $J^1(M, N) = \text{Hom}(\pi_M^* TM, \pi_N^* TN)$, where π_M, π_N are the natural projection maps of the product $M \times N$ on to its factors.

Whenever $k < l$, we can “forget higher order terms” to get a k -jet from an l -jet; i.e. there are natural projections $J^l \rightarrow J^k$, since l^{th} order contact implies k^{th} order contact. When $r \geq 1$, we write $J_{\text{inv}}^r(M, N)$ for the open submanifold of $J^r(M, N)$ consisting of jets with invertible 1-jet part.

2.3 Parabolic PDE and Maximum Principles

While our motivations may come from differential geometry and topology, in order to prove anything about geometric flows we will need a lot of analytic background. Indeed the reason we choose geometric evolution equations that are analogous to the heat equation

is precisely so that we can recover some of the nice properties of the latter, which we do by applying ideas from PDE theory. In this section we will quickly define parabolic partial differential equations on manifolds and prove some maximum principles, which are one of the most elementary yet powerful tools for studying solutions of such PDE.

For readers unfamiliar with the theory of elliptic and parabolic PDE, we recommend [Eva98] to get a grasp on the ideas and techniques and [Tay96] to see how similar theory can be developed in the setting of Riemannian manifolds. For a more comprehensive reference one can see the standard reference for elliptic equations [GT83] and its parabolic companion [Lie96].

2.3.1 Definitions and Notation

We will now briefly define (scalar) parabolic partial differential equations on a Riemannian manifold M (possibly with boundary). Throughout this section, all functions and sections are allowed to be rough - by default, we assume measurability and nothing more.

Definition 2.3.1. A *quasilinear elliptic (partial differential) operator* is an operator E sending twice continuously differentiable functions to continuous functions of the form

$$(Eu)(x) = a^{ij}(x, u(x), du(x)) \nabla_i \nabla_j u(x) + b(x, u(x), du(x)) \quad (2.3.1)$$

where $a : \mathbb{R}_M \oplus T^*M \rightarrow \text{Sym}^2 TM$ is an *elliptic* fibre-preserving bundle map, where elliptic means $a^{ij}(x, z, p) v_i v_j > 0$ for all non-zero $v \in T^*M$. In particular this implies there are functions $\lambda(x, z, p), \Lambda(x, z, p)$ such that $\lambda(x, z, p) |v|^2 \leq a(v, v) \leq \Lambda(x, z, p) |v|^2$. If λ has a positive lower bound λ_0 then we say the operator is *uniformly elliptic* with constant λ_0 . If we only have $a^{ij} v_i v_j \geq 0$ (i.e. $\lambda_0 = 0$) then we say E is *weakly elliptic*.

If a^{ij} is a function of x alone (i.e. $a \in \Gamma(\text{Sym}^2 TM)$) then we say the operator is *semilinear*; and if $b = b^i \nabla_i u + cu$ for some $b \in \Gamma(TM)$, $c : M \rightarrow \mathbb{R}$ then it is *linear*:

$$(Eu)(x) = a^{ij}(x) \nabla_i \nabla_j u(x) + b^i(x) \nabla_i u(x) + c(x) u(x). \quad (2.3.2)$$

When using the theory of weak solutions, it is more convenient to work with operators written in *divergence form*

$$(Eu)(x) = \nabla_i (a^{ij}(x) \nabla_j u(x)) + b^i(x) \nabla_i u(x) + c(x) u(x). \quad (2.3.3)$$

While an operator in general form can always formally be rewritten as one in divergence form (with first-order coefficient $b^i - \nabla_j a^{ij}$), note that we lose some regularity in the coefficients by doing so: in order to ensure a given regularity of the coefficients of this resulting operator we need the same regularity assumption on the *derivatives* of the original a^{ij} . Thus the optimal regularity theory for equations in general form cannot be obtained by this kind of transformation; so (as we will see in section 2.4.5) historically the divergence-form equations were better understood, but as time passed many very similar estimates

were obtained for equations in general form.

Definition 2.3.2. A (weakly) parabolic operator P on $M \times [0, T)$ is one of the form $Pu(x, t) = \partial_t u(x, t) - (E(t)u(\cdot, t))(x)$ where each $E(t)$ is a (weakly) elliptic operator on the slice $M \times \{t\} \simeq M$. Such an operator is *uniformly parabolic* with constant λ if every $E(t)$ is uniformly elliptic with constant λ , and *autonomous* if E is constant in t . A *parabolic (partial differential) equation* is one of the form $Pu = f$. We say u is a *subsolution* (*supersolution*) of $Pu = f$ if $Pu \leq f$ ($Pu \geq f$).

In this time-dependent setting we will use the shorthand $X = (x, t)$ and similarly $X_0 = (x_0, t_0)$, etc, for space-time coordinates.

Many standard results and estimates for elliptic equations are easily extended to parabolic ones, but with local neighbourhoods replaced by “local history” - that is, we replace a ball with a cylinder back in time:

Definition 2.3.3. For $X_0 = (x_0, t_0) \in M \times [0, T)$, the *parabolic neighbourhood* $Q(X_0, R)$ is the set of points (x, t) such that $x \in B(x_0, R)$ and $t \in (t_0 - R^2, t_0)$.

Note the timescale R^2 - a recurring theme in the study of parabolic equations is that times correspond to squared lengths. (The classic illustration of this is scaling symmetry: $u(\lambda x, \lambda^2 t)$ solves the heat equation whenever $u(x, t)$ does.) Related to this is the notion of parabolic distance:

Definition 2.3.4. The *parabolic distance* between two points $X_1, X_2 \in M \times [0, T)$ is defined by $d_P(X_1, X_2) = \sup \left(d(x_1, x_2), |t_1 - t_2|^{1/2} \right)$.

Note that the parabolic neighbourhoods are actually strict subsets of the metric balls of parabolic distance - we have $Q(X_0, R) = \{X : d_P(X_0, X) \leq R, t < t_0\}$. This time-asymmetry is characteristic of parabolic equations, which are very well-behaved running forwards but quite pathological backwards. This time-directedness shows up in the parabolic idea of the boundary of a domain, too:

Definition 2.3.5. The *parabolic boundary* $\mathcal{P}\Omega$ of an open set $\Omega \subset M \times [0, \infty)$ is the set of points $X \in \bar{\Omega}$ such that $Q(X, R)$ intersects $\Omega^c \cup \partial\Omega$ for every $R > 0$. For a domain $D \subset M$, the parabolic boundary of the cylinder $\Omega = D \times (0, T)$ is the union of the *side* $\mathcal{S}\Omega = \partial D \times (0, T)$, the *bottom* $\mathcal{B}\Omega = D \times \{0\}$ and the *corner* $\mathcal{C}\Omega = \partial D \times \{0\}$.

The reason for this definition is by analogy to elliptic boundary value problems: we expect to establish a unique solution to a parabolic PDE on Ω by prescribing the value on the parabolic boundary. Here this corresponds to choosing an initial condition (prescribing on $\mathcal{B}\Omega$) and a boundary condition (prescribing on $\mathcal{S}\Omega$), and thus we call such a problem an initial-boundary value problem (or *Cauchy-Dirichlet*) problem. The next section will make much use of the parabolic boundary in this role.

2.3.2 Maximum Principles

Now that we have established our definitions, we move on to proving maximum principles. These are all variants of the following claim: if u is a solution of a second-order parabolic equation (of the right structure), then the maximum (or minimum, or something else analogous) of u must be attained on the parabolic boundary. The more physical way to think about this is that the equation *preserves* bounds on the initial data, assuming the bounds are not violated on the boundary. This kind of result will provide us with many comparison arguments, uniqueness results, and solution estimates from here forward. Before we begin with maximum principles proper, we will start with some classical comparison results for ODEs, which will get our toes wet with some easy calculations and serve as a nice comparison with the PDE results to come.

Proposition 2.3.6 (Gronwall's Inequality). *If $u, v : [0, T] \rightarrow \mathbb{R}$ satisfies $u'(t) \leq \beta(t)u(t)$, then for all $t > 0$, $u(t)$ is bounded from above by the solution of $v'(t) = \beta(t)v(t)$ with $v(0) = u(0)$.*

Proof. Note that we can write the exact solution

$$v(t) = u(0) \exp \int_0^t \beta.$$

Defining $q(t) = u(t) / \exp \int_0^t \beta$, we compute

$$q'(t) = \frac{u'(t) - \beta(t)u(t)}{\left(\exp \int_0^t \beta\right)^2} \exp \int_0^t \beta \leq 0;$$

so the initial condition $q(0) = u(0)$ implies $q(t) \leq u(0)$ at later times; i.e. $u(t) \leq v(t)$ as desired. \square

Proposition 2.3.7 (Nonlinear comparison principle). *If $F : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Lipschitz condition*

$$|F(x) - F(y)| \leq M|x - y|$$

and $u, v : [0, T] \rightarrow \mathbb{R}$ satisfy $u'(t) \leq F(u)$, $v'(t) \geq F(v)$ and $u(0) \leq v(0)$, then $u(t) \leq v(t)$ for all $t \geq 0$.

Proof. Let $w(t) = u(t) - v(t)$, so $w(0) \leq 0$ and $w'(t) \leq F(u) - F(v)$. Assume for the sake of contradiction that there is some time t_1 at which $u(t_1) > v(t_1)$, and let $t_0 < t_1$ be the last time at which $u(t_0) \leq v(t_0)$. On the interval $[t_0, t_1]$ we have $w \geq 0$, so using the Lipschitz condition we can estimate

$$w'(t) \leq |F(u) - F(v)| \leq M|u - v| = Mw(t).$$

Gronwall's inequality then yields $w(t) \leq w(t_0)e^{M(t-t_0)} = 0$ for all $t \in [t_0, t_1]$, a contradiction. \square

For the next few results, we will let

$$Pu = \partial_t u - a^{ij} \nabla_i \nabla_j u - b^j \nabla_j u - cu$$

be a linear parabolic operator. However, one should note the maximum principles for linear and semilinear equations are frequently useful for studying quasilinear equations, since a solution u to a quasilinear PDE is also a solution to the (semi)linear PDE obtained by “freezing” (some of) the coefficients at u , and there are very few regularity requirements on the coefficients for maximum principles to work. For simplicity we will work on a cylinder $\Omega = D \times [0, T)$. We start with a soft version:

Lemma 2.3.8 (Weakest maximum principle). *If u satisfies $Pu < 0$ on Ω and $u < 0$ on $\mathcal{P}\Omega$, then $u < 0$ on Ω .*

Proof. Assume otherwise. Letting $t_0 = \inf \{t : u(x, t) \geq 0 \text{ for some } x\}$, then by continuity there must be a point x_0 such that $u(x_0, t_0) = 0$. By our construction of this point we know that $u(x, t) < 0$ for all x and all $t \leq t_0$, and thus we have $\partial_t u(x_0, t_0) \geq 0$, $du(x_0, t_0) = 0$ and $\nabla^2 u(x_0, t_0) \leq 0$. In particular the restriction of $u(\cdot, t_0)$ to any geodesic through x_0 has a maximum at x_0 , so choosing a geodesic normal coordinate system to diagonalize $a^{ij}(x_0)$ we get

$$a^{ij}(x_0) \nabla_i \nabla_j u(x_0) \leq \sum_{i=1}^n \lambda_i(x_0) \frac{d^2}{ds^2} \Big|_{s=0} u(x_0 + se_i) \leq 0.$$

Thus we have $Pu(x_0) \geq 0$, contradicting our assumption. \square

We can use this along with a very simple approximation argument to get a version that applies to solutions rather than just strict subsolutions:

Theorem 2.3.9 (Weak maximum principle). *If u satisfies $Pu \leq 0$ on Ω and $u \leq 0$ on $\mathcal{P}\Omega$, then $u \leq 0$ on Ω .*

Proof. For $\epsilon > 0$, define $u_\epsilon(x, t) = u(x, t) - \epsilon(1 + t)$ and note that $u_\epsilon < u \leq 0$ on $\mathcal{P}\Omega$ and $Pu_\epsilon = Pu - \epsilon < 0$ on Ω . Applying Lemma 2.3.8 to u_ϵ yields $u_\epsilon \leq 0$; i.e. $u(x, t) \leq \epsilon(1 + t)$ for all $\epsilon > 0$, which implies $u \leq 0$. \square

Applying this to the difference of two solutions gives:

Corollary 2.3.10 (Comparison principle). *If $Pu \leq Pv$ on $\mathcal{P}\Omega$ and $u \leq v$ on $\mathcal{P}\Omega$ then $u \leq v$ on Ω .*

We would like to be able to preserve more general bounds than just $u \leq 0$. Letting $v = M$ be a constant we see that $Pv = cM$; so the comparison principle gives:

Corollary 2.3.11 (Weak maximum principle, generalised version). *If $Pu \leq 0$ on Ω , $u \leq M$ on $\mathcal{P}\Omega$ and $cM \leq 0$ then $u \leq M$ on Ω .*

So if $c \geq 0$ we can preserve non-positive upper bounds, while if $c \leq 0$ we can preserve non-negative upper bounds. More generally again, we can compare to a first-order ODE. For linear equations where the coefficient c is bounded above by C , this is easily obtained by applying the maximum principle to $e^{-Ct}u$:

Corollary 2.3.12 (Weak maximum principle, exponential bound). *If $c \leq C$, $Pu \leq 0$ on Ω and $u \leq Me^{Ct}$ on $\mathcal{P}\Omega$ then $u \leq Me^{Ct}$ on Ω .*

For a stronger comparison principle we shall consider semilinear operators: suppose for the rest of this section that $Pu = \partial_t u - a^{ij}\nabla_i\nabla_j u - b^i\nabla_i u - F(u)$ with $F : \mathbb{R} \rightarrow \mathbb{R}$ Lipschitz.

Proposition 2.3.13 (Semilinear comparison principle). *If $Pu \leq Pv$ on Ω and $u \leq v$ on $\mathcal{P}\Omega$ then $u \leq v$ on Ω .*

Proof. As in the proof of Theorem 2.3.9, it is easy to establish the analogous result for strict inequalities: at the first point where $u \geq v$ we would have $u = v$ and thus $F(u) = F(v)$, along with the usual $\partial_t u \geq \partial_t v$ and $a^{ij}\nabla_i\nabla_j(u - v) \leq 0$ and $b^i\nabla_i(u - v) = 0$. Putting this all together contradicts $Pu < Pv$.

To extend this to require only the weak inequalities we are actually assuming, let $u_\epsilon(x, t) = u(x, t) - \epsilon e^{2Ct}$ where C is the Lipschitz constant of F . Then we have $u_\epsilon < v$ on $\mathcal{P}\Omega$ and

$$Pu_\epsilon = Pu - 2C\epsilon e^{2Ct} - F(u + \epsilon e^{2Ct}) + F(u) \leq Pu - 2C\epsilon e^{2Ct} - C\epsilon e^{2Ct} < Pu \leq Pv,$$

so the strict version gives $u_\epsilon < v$ for every ϵ , and thus $u \leq v$. \square

When v depends only on t this strengthens Corollary 2.3.12 to comparison with the ODE part of any semilinear equation:

Corollary 2.3.14 (Comparison with the corresponding ODE). *Assume $u(x, t)$ satisfies $Pu \leq 0$ on Ω . If $v(t)$ is a solution to the ODE $v'(t) = F(v(t))$ such that $u(x, t) \leq v(t)$ on $\mathcal{P}\Omega$, then $u(x, t) \leq v(t)$ on Ω .*

This is particularly useful when $\Omega = M \times [0, T)$ for a boundaryless manifold M , where it tells us that the supremum of u grows at most as fast as if it was a solution of the corresponding ODE.

Remark. When applying these maximum principles, we will often reason “using the proof”: for example, suppose we want a maximum principle for some function u satisfying the inequality $Pu \geq |\nabla u|^2$. We could obtain this by applying one of the above theorems to the modified operator $P'_u v = Pv - \langle \nabla v, \nabla u \rangle$ (so that $P'_u u \geq 0$); but instead we can simply note that $Pu \geq 0$ whenever $\nabla u = 0$, so the proof goes through. Thus when applying a maximum principle to show that some inequality $u \geq v$ is preserved, we simply need to show that $Pu(x) \geq Pv(x)$ under the zeroth-order condition $u(x) = v(x)$ and the first-order condition $\nabla u(x) = \nabla v(x)$.

2.4 Function Spaces and Standard Estimates

Picking up a random paper or text in PDE theory, one could almost surely be forgiven for thinking that they're reading an encyclopaedia of inequalities. Indeed almost all of the work in any recent PDE result is in the estimates, whether they will be used for existence, regularity or simply understanding the behaviour of solutions. These estimates are most often simply on the norm of the solution in some normed space, with the choice of norm often being key. For example norms based on integrals (such as the L^2 -based Sobolev norms that we will define shortly) are an extremely powerful tool for establishing the existence of solutions to linear equations; but for non-linear equations one is often better off working with a Hölder norm (although L^p norms are still very useful, often with p taken to infinity to get uniform control).

This section includes definitions of the function spaces we are interested in along with standard results about them, including both universal inequalities and estimates that hold only for solutions of certain PDE. All of these definitions and results (or their natural analogues) are very well known in the case of a bounded domain in Euclidean space. The generalization to the compact manifold setting is technical but not problematic. For the reader who is not familiar with normed spaces in PDE theory (e.g. one who wonders how these estimates are derived and how they are applied) I recommend [Bre10] or [Eva98].

I've chosen to include a complete proof that Hölder spaces over compact manifolds are independent of the metric(s) used to define them - this is a very well-known fact that is not difficult to prove, but (at least in my experience) it is difficult to find a complete proof in the literature. The equivalences are stated below, but the proofs have been extracted to Appendix A. Similar results for the Sobolev spaces are omitted, but true - see e.g. [Heb96].

We shall define everything on a **compact** Riemannian m -manifold (M, g) with geodesic distance function d , Riemannian volume measure μ and Levi-Civita connection ∇ , and will often use g and μ implicitly - for example $\int |V|^2$ really means $\int_M g(V, V) d\mu$.

Since this is the first section in which we will be interested in estimates, there is some notation to be aware of: $A \lesssim B$ means that $A \leq CB$ for some constant C independent of A and B : for example $\|f\|_2 \lesssim \|Df\|_2$ means there is some constant C such that $\|f\|_2 \leq C \|Df\|_2$ for all f under consideration. The notation C in particular will denote a constant that may change from line to line. Recall that two norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent if $\|f\|' \lesssim \|f\|$ and $\|f\| \lesssim \|f\|'$. For completeness, we will start by stating some elementary inequalities that we will be using all over the place:

- For any inner product space $(V, \langle \cdot, \cdot \rangle)$ we have the *Cauchy-Schwarz inequality* $\langle v, w \rangle^2 \leq \langle v, v \rangle \langle w, w \rangle$.
- For any real numbers a, b we have *Young's inequality* $|ab| \leq \frac{1}{2}(a^2 + b^2)$.
- Generalizing Young's inequality, for any real numbers a, b and any $\epsilon > 0$ we have the *Peter-Paul inequality* $|ab| \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$.

While all three of these are indispensable, the Peter-Paul inequality is strikingly powerful for such a simple trick: when we have a very small but good term a^2 , it will allow us to absorb arbitrarily large terms that are linear in a at the cost of adding a constant.

2.4.1 C^k Spaces

The simplest topology to put on a space of functions is that of uniform convergence:

Definition 2.4.1. $C^0(M) = C^{0,0}(M)$ is the Banach space of continuous functions $M \rightarrow \mathbb{R}$, with norm $|f|_0 = |f|_{0,0} = \|f\|_\infty = \sup f$.

Since we are working with differential equations, we often need to bound the distance between derivatives of functions; so we are also interested in the C^k norms:

Definition 2.4.2. $C^k(M)$ is the Banach space of k -times continuously differentiable functions $M \rightarrow \mathbb{R}$, with norm $|f|_k = |f|_{k,0} = \sum_{i=0}^k \left| |\nabla^i f|_g \right|_0$.

Alternatively we could write $\sum_{|I| \leq k} \left| |\nabla_I f|_g \right|_0$, where $|I|$ is the number of indices in the multi-index I .

When we want to be explicit about the domain (and maybe the metric) we are using to define the norm, we will write this as $|f|_{k,0;M}$ or $|f|_{k,0;(M,g)}$.

For the purposes of a nice coordinate-free definition we have used the iterated covariant derivatives $\nabla^i f$ and the metric norm $|\cdot|_g$, so this norm depends on the metric we choose for M ; but it turns out that in the compact case, different metrics generate equivalent norms:

Proposition 2.4.3. If g, h are Riemannian metrics on M , then the C^k norms $|\cdot|_{k,0;(M,g)}$ and $|\cdot|_{k,0;(M,h)}$ that they generate are equivalent.

Proof. See §A.1. □

In particular, by modifying the metric to be flat in an injective neighbourhood of a given point, we see that local C^k estimates with respect to the Riemannian geometry are equivalent to local C^k estimates in normal coordinates.

If f is instead a section of a vector bundle E equipped with a fibre metric ∇^E and compatible connection, we can generalize the above definitions in an obvious way: take iterated derivatives of f using the natural connection induced on the bundles $(T^*M)^{\otimes i} \otimes E$ by the Levi-Civita connection and ∇^E , and then measure the results using the natural metric induced by g and the fibre metric.

The space of smooth functions is the decreasing intersection $C^\infty = \bigcap_{k \in \mathbb{Z}} C^k$, which is a Fréchet space when topologized using the countable family of C^k norms. Explicitly, the topology can be defined by the complete translation-invariant metric

$$d(f, g) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{|f - g|_k}{1 + |f - g|_k},$$

so that $f_j \rightarrow f$ in C^∞ if and only if $f_j \rightarrow f$ in every C^k . Since the topology of a metric space is determined by its convergent sequences, we can forget about the metric and just remember this sequential characterization of smooth convergence. Since we are working with compact manifolds, there is no need to distinguish between the strong and weak C^∞ topologies.

2.4.2 Hölder Spaces

Some of the most powerful results in the theory of classical solutions for elliptic and parabolic PDE are in terms of the Hölder spaces, which are a strengthening of the C^k spaces requiring a power-law estimate on the modulus of continuity of the k^{th} derivative. As before, we will define them here on a compact Riemannian manifold M .

Definition 2.4.4. For $\alpha \in (0, 1]$ and $f \in C^0(M)$, define the Hölder α -seminorm by

$$[f]_\alpha = \sup_{x, y \in M} \frac{|u(x) - u(y)|}{d(x, y)^\alpha}.$$

If $\xi \in C^0(M, E)$ is instead a section of a bundle E equipped with a metric h and compatible connection ∇^E , we want to analogously define the seminorm by

$$[\xi]_\alpha = \sup_{x, y \in M} \frac{d'(\xi(x), \xi(y))}{d(x, y)^\alpha}.$$

Here the correct notion of “distance” $d'(\xi(x), \xi(y))$ between two vectors at different points is not so clear. The obvious idea is to define $d'(\xi(x), \xi(y)) = |\xi(x) - \tau_\gamma \xi(y)|_h$ where γ is a minimizing geodesic joining x to y , where $\tau_\gamma : E_x \rightarrow E_y$ denotes the ∇^E -parallel translation map along γ ; but when x, y are far enough apart minimizing geodesics are not necessarily unique. We can remedy this, however - short enough geodesics are unique, and (as with all continuity estimates) we really care only about the limit as $x \rightarrow y$. To make this precise, let ι_g denote the injectivity radius of (M, g) , which is positive because M is compact [Kli82, Proposition 2.1.10].

Definition 2.4.5. For $\alpha \in (0, 1]$ and $\xi \in C^0(M, E)$, define the Hölder α -seminorm by

$$[\xi]_\alpha = \sup_{\substack{x \neq y \\ d(x, y) < \iota_g}} \frac{|\xi(x) - \tau_{\gamma_{yx}} \xi(y)|_h}{d(x, y)^\alpha}$$

where γ_{yx} is the (unique!) minimizing geodesic joining y to x . If we want to make the geometric structure we are using explicit we can write $[\xi]_{\alpha; g, h, \nabla^E}$.

Definition 2.4.6. The set of C^k functions with $|f|_k + [\nabla^k f]_\alpha < \infty$ is the *Hölder space* $C^{k, \alpha}(M)$, which is a Banach space when equipped with the *Hölder norm* $|f|_{k, \alpha} = |f|_{k, 0} + [\nabla^k f]_\alpha$. For $k = 0$, $\alpha = 1$ this is the *Lipschitz space/norm*. When we want to make the domain/metric explicit we will write $|f|_{k, \alpha; (M, g)}$ or $|f|_{k, \alpha; g}$.

As with the C^k norm, the exact values of this norm depend upon the geometry defined by g , but in the compact case the Hölder space itself turns out not to. For brevity the details have been left to §A.2, where we give an alternate characterization of $C^{k,\alpha}(M)$ using an atlas instead of a metric, and consequently derive the following result:

Proposition 2.4.7. *If g, h are two Riemannian metrics on M , then the $C^{k,\alpha}$ norms $|\cdot|_{k,\alpha;g}$ and $|\cdot|_{k,\alpha;h}$ are equivalent.*

In particular, if s is a tensor field and $R_0 > 0$ is fixed, a Hölder estimate $|s^I|_{k,\alpha} \leq H$ for the components of s in an arbitrary geodesic normal coordinate system of radius R_0 (with H independent of the choice of coordinate origin) implies $|s|_{k,\alpha} \leq CH$ for some constant C depending only on R_0 and M . Thus we can obtain global Hölder estimates on manifolds simply by proving a *uniform* local Hölder estimate in normal coordinates.

2.4.3 L^p and Sobolev Spaces

While we will not be needing the theory of weak solutions for (the interesting parts of) our results, integral estimates will still be very useful for determining the long-time behaviour of solutions. Thus we will quickly define these normed spaces and state the inequalities we will need. For a more detailed development of this theory on compact manifolds, see [Aub98] or [Heb96].

We start with the simplest building block, the L^p spaces.

Definition 2.4.8. For $p \in [1, \infty)$, the pre- L^p space $L_-^p(M)$ is the set of functions $M \rightarrow \mathbb{R}$ such that $\int |f|^p d\mu < \infty$. The L^p space is then defined as $L^p(M) = L_-^p(M) / \sim$ where $f \sim g$ if and only if $f = g$ μ -almost everywhere. The L^p norm of $[f]$ is defined by $\|[f]\|_p^p = \int |f|^p d\mu$, which is well-defined since $\int |f|^p d\mu = \int |g|^p d\mu$ whenever $f = g$ almost everywhere. The essentially bounded functions $L^\infty(M)$ are defined in the same fashion by replacing $\int |f|^p d\mu$ with $\text{ess sup } |f|$.

We will always conflate L^p “functions” (equivalence classes) with their representatives, so we will write just $\|f\|_p$ instead of $\|[f]\|_p$.

The L^p spaces are quite nice to work with - they are all Banach spaces, for $p \in (1, \infty)$ they are reflexive, and for $p = 2$ even Hilbert (with inner product $\langle u, v \rangle_2 = \int uv d\mu$). We could alternatively have defined $L^p(M)$ as the completion of the compactly supported smooth functions $C_c^\infty(M)$ under the norm $\|\cdot\|_p$, since continuous functions are unique in their L_-^p classes and C_c^∞ is dense in L^p .

The most fundamental result for L^p spaces is Hölder’s inequality:

Proposition 2.4.9 (Hölder’s Inequality). *If $f \in L^p$, $g \in L^{p'}$ and $1/p + 1/p' = 1/r$, then $fg \in L^r$ with norm $\|fg\|_r \leq \|f\|_p \|g\|_{p'}$.*

Since we are assuming M is compact and thus has finite volume V , applying this to $f \in L^q$ with $p \leq q \leq \infty$ gives

$$\|f\|_p \leq \|f\|_q \|1\|_{pq/(q-p)} = V^{(q-p)/pq} \|f\|_q,$$

so $L^p \hookrightarrow L^q$; i.e. $L^p \subset L^q$ and the inclusion map is bounded. For $f \in L^\infty$ we have $\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty$ and thus our most elementary method to pass from integral estimates to uniform ones: if a continuous function f has $\|f\|_p \leq M$ for all finite p , then the same estimate holds for the C^0 norm.

To define the Sobolev spaces we have two conventional options: either develop the notion of weak derivatives in the Riemannian setting, or simply complete C_c^∞ under the Sobolev norm. For brevity we will go with the latter approach; but for those familiar only with the theory on \mathbb{R}^n we stress that weak derivatives work identically here.

Definition 2.4.10. For $k \in \mathbb{N}$ and $p \geq 1$, the Sobolev space $W^{k,p}(M)$ is the completion of $C_c^\infty(M)$ under the norm $\|f\|_{k,p}^p = \sum_{j=0}^k \left\| |\nabla^j f|_g \right\|_p^p$.

Sobolev spaces are also Banach; and for $p = 2$ they are Hilbert (with inner product $\langle u, v \rangle_{k,2} = \sum_{j=0}^k \langle \nabla^j u, \nabla^j v \rangle_2$). As with Hölder spaces, the Sobolev norms on a Riemannian manifold depend on the metric we choose, but for compact manifolds the choice of metric does not change the Sobolev spaces themselves - the norms induced by different metrics will always be equivalent. As in the case of Hölder spaces (see §A.2), we could choose instead to define these spaces using coordinate charts instead of the Riemannian structure: if $A = (\mathcal{U}_i, \varphi_i)$ is an atlas satisfying the hypotheses of Proposition A.2.3 and η_i a subordinate partition of unity, then the norm defined by

$$\|f\|_{k,p;A}^p := \sum_i \left\| (\eta_i f) \circ \varphi_i^{-1} \right\|_{k,p;\varphi(\mathcal{U}_i)}^p$$

is equivalent to $\|\cdot\|_{k,p}^p$, which allows us to easily transfer most of the following results directly from \mathbb{R}^n to compact manifolds.

Now to one of the most fundamental inequalities for Sobolev spaces over compact domains, which bounds the mean oscillation of a function in terms of the L^2 norm of its derivative.

Proposition 2.4.11 (Poincaré Inequality). *There is some constant C such that*

$$\|u - \bar{u}\|_2 \leq C \|\nabla u\|_2$$

for all $u \in W^{1,2}(M)$, where $\bar{u} = f_M u = (\int_M u) / \int_M 1$ is the average of u .

Remark. Recalling the variational characterization of the Laplace eigenvalues, we see that the constant C appearing in the Poincaré inequality is in fact the reciprocal of the first eigenvalue of M .

Next we state a very general interpolation inequality for Sobolev norms, which holds either on a compact manifold or \mathbb{R}^m [Aub98, Thm 3.70] (but not non-compact manifolds in general).

Proposition 2.4.12 (Gagliardo-Nirenberg interpolation inequality). *If*

$$\frac{1}{p} = \frac{j}{m} + \left(\frac{1}{r} - \frac{k}{m} \right) \alpha + \frac{1-\alpha}{q}$$

and $j/k \leq \alpha \leq 1$, then there is a constant C such that

$$\|\nabla^j u\|_p \leq C \left\| \nabla^k u \right\|_r^\alpha \|u\|_q^{1-\alpha}.$$

Taking $\alpha = 1$ we recover the first part of the *Sobolev embedding theorem*:

Corollary 2.4.13. *If $j \leq k$ and*

$$\frac{1}{p} = \frac{1}{r} + \frac{j-k}{m}$$

then $W^{k,r}(M)$ embeds continuously in to $W^{j,p}(M)$.

If we require strict inequality then the embedding becomes compact:

Proposition 2.4.14 (Rellich-Kondrachov embedding theorem). *If $j < k$ and $k - m/r > j - m/p$ (so in particular when $p = r$) then the embedding $W^{k,r} \hookrightarrow W^{j,p}$ is compact.*

The last part of the Sobolev embedding theorem is *Morrey's inequality*, the key to much of the regularity theory for elliptic and parabolic equations:

Proposition 2.4.15. *If $m < p \leq \infty$ then there is a constant C such that*

$$|u|_{0,\alpha} \leq C \|u\|_{1,p},$$

where $\alpha = 1 - \frac{m}{p}$.

In light of this inclusion $W^{1,p} \hookrightarrow C^{0,\alpha}$, we can start with a weak solution (i.e. a function u in some Sobolev space satisfying some system of integral equations) and establish that in fact the solution is C^k (and thus a strong solution if k is large enough) simply by obtaining an estimate for its norm in some $W^{k+1,p}$ space.

2.4.4 Parabolic Function Spaces

In the preceding sections we have defined norms for functions $M \rightarrow \mathbb{R}$, which gives us the kind of spaces we need to study elliptic equations; but if we want to study equations that treat time specially, then we naturally need spaces that treat time specially. Here we will very quickly define the parabolic versions of Sobolev and Hölder spaces over the domain $M \times [0, T)$, with the recurring moral theme being that t is like x^2 - recall this trend from section 2.3.1. One important point to keep in mind is that whenever f satisfies a certain bound in one of these parabolic spaces, each time slice of f satisfies the same bound in the time-independent version.

Definition 2.4.16. The space $C^k(M \times [0, T])$ is defined by the norm

$$|f|_{k;0,T} = \sum_{|I|+2j \leq k} \sup_{M \times [0,T]} \left| \nabla^I \partial_t^j f \right|_g.$$

So the parabolic C^0 , C^1 norms are simply the time supremum of the corresponding spatial norms; but at C^2 we add in the supremum of the first time derivative. Thus the (local) parabolic C^2 space is where we expect to find classical solutions of parabolic equations.

Definition 2.4.17. For $\alpha \in (0, 1]$, the parabolic Hölder space $C^{k,\alpha}(M \times [0, T])$ is defined by the norm

$$|f|_{k,\alpha;0,T} = |f|_{k;0,T} + \sum_{|I|+2j=k} \left[\nabla^I \partial_t^j f \right]_\alpha$$

where the parabolic Hölder seminorm is

$$[\xi]_\alpha = \sup_{X \neq Y} \frac{|\xi(X) - \xi(Y)|}{d_P(X, Y)^\alpha}.$$

For $k > 0$ the argument to the seminorm is a tensor, so the difference in the numerator needs to be interpreted in the sense of 2.4.5.

One useful characterization of the Hölder seminorm is the localized version: we say ξ is α -Hölder at X with seminorm C if $\text{osc}_{Q(X,R)} \xi \leq CR^\alpha$ for all R less than some $R_0 > 0$, where we define $\text{osc}_\Omega \xi = \sup_{X,Y \in \Omega} |\xi(X) - \xi(Y)|$ using parallel transport on small enough sets Ω . If ξ is locally α -Hölder at each point X with the same seminorm, then it is uniformly α -Hölder with the same seminorm.

Definition 2.4.18. The parabolic Sobolev space $W^{k,p}(M \times [0, T])$ is defined by the norm

$$\|f\|_{k,p;0,T}^p = \sum_{|I|+2j \leq k} \int_0^T \int_M \left| \nabla^I \partial_t^j f \right|^p dx dt.$$

Note that these norms are genuine generalizations of their time-independent counterparts: if we define $\hat{f}(x, t) = f(x)$ then we get $|\hat{f}|_{k,\alpha;0,T} = |f|_{k,\alpha}$ and $\|\hat{f}\|_{k,p;0,T} = T^{1/p} \|f\|_{k,p}$. As in the time-independent case, these norms are defined in terms of a metric, but so long as M is compact they in fact define topological vector spaces that are independent of the metric used.

2.4.5 Standard Parabolic Estimates

In addition to the ubiquitous Sobolev embeddings, we will need a few standard Hölder estimates for solutions of parabolic PDE. We will state most of them for operators on \mathbb{R}^m - these will suffice for our purposes, since (as we have seen earlier) Hölder estimates for a function u on a compact manifold are equivalent to Hölder estimates for the composition of u with nice enough charts.

The first is the interior Schauder estimate, which gives Hölder estimates for the second derivatives of solutions to equations with Hölder continuous coefficients and data. Many variants of this estimate have been proved in countless ways since its conception in the 1930s; in general they take the form $|u|_{2,\alpha} \leq C \left(|u|_0 + |\text{data}|_{0,\alpha} \right)$, where the data includes Pu along with any initial/boundary data.

Since we will be dealing with maps between manifolds, we will need the Schauder estimate for systems. The following can be found with proof in e.g. [Bak11, Prop 3.18] or [Sch96, Theorem 3].

Proposition 2.4.19 (Interior Schauder Estimate). *Let $u : Q(0, R) \rightarrow \mathbb{R}^k$ be a solution of the parabolic system*

$$\partial_t u^\alpha - a_\beta^{\alpha ij} \partial_i \partial_j u^\beta - b_\beta^{\alpha i} \partial_i u^\beta - c_\beta^\alpha u^\beta = f^\alpha$$

where $a_\beta^{\alpha ij}$ satisfies a uniform Legendre-Hadamard ellipticity condition

$$\lambda |\xi|^2 |v|^2 \leq a_\beta^{\alpha ij} \xi_i \xi_j v^\alpha v^\beta \leq \Lambda |\xi|^2 |v|^2$$

and the symmetry condition $a_\beta^{\alpha ij} = a_\alpha^{\beta ji}$. If the coefficients a, b, c are all $C^{0,\gamma}$ for some $\gamma > 0$, then for any $\mu \in (0, 1)$ there is a constant C depending only on $\lambda, \Lambda, \mu, n, k, \gamma$ and the $C^{0,\gamma}$ norms of the coefficients such that

$$|u|_{2,\gamma;Q(0,\mu R)} \leq C \left(|f|_{0,\gamma} + R^{-2m-\gamma} |u|_0 \right).$$

While this estimate is useful, when studying nonlinear equations we often do not have such strong control on the coefficients; so we need a Hölder estimate with weaker assumptions. Such a result was first obtained independently by Nash [Nas58] and de Giorgi [DG57], with the only assumption on a^{ij} being uniform ellipticity. We will derive this estimate from a weak Harnack inequality, which bounds a positive supersolution from below in terms of its average at a slightly earlier time. Inequalities of this type have been used for Hölder estimates at least as far back as Moser [Mos64], and were later developed by Trudinger [Tru68], Krylov and Safonov [KS80], and Gruber [Gru84] for equations in more general form. We assume $b^i = c = 0$ for simplicity - for more general statements and proofs, see Chapters VI and VII of [Lie96]. This result holds (with slight differences) for both general and divergence-form operators.

Lemma 2.4.20 (Weak Harnack Inequality). *Let P be a linear parabolic operator $\partial_t - a^{ij} \partial_i \partial_j$ or $\partial_t - \partial_i (a^{ij} \partial_j \cdot)$ and define $\Theta(X, R) = Q((x, t - 4R^2), R)$. Then are positive constants C and p depending only on n, λ, Λ such that*

$$\left(\int_{\Theta(X,R)} u^p \right)^{1/p} \leq C \left(\inf_{Q(X,R)} u + R^{n/(n+1)} \|f\|_{n+1;4R} \right)$$

for all positive $u \in W^{2,n+1}(Q(X, 4R))$ satisfying $Pu \geq f$. Here $\int_\Theta u = \frac{1}{\mu(\Theta)} \int_\Theta u$ denotes

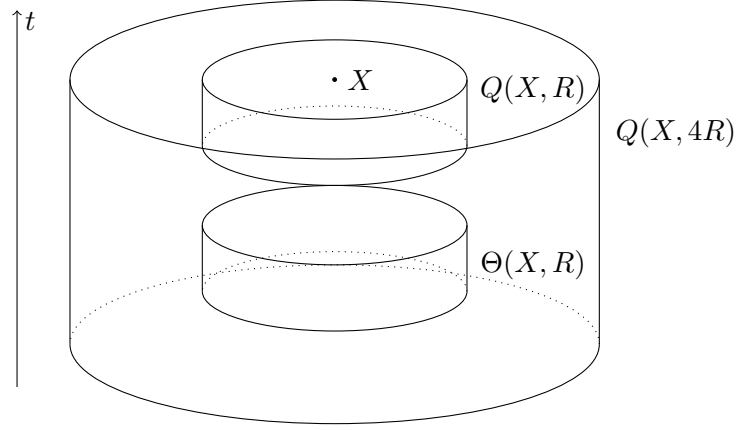


Figure 2.4.1: Illustration of the space-time cylinders $Q(X, R)$, $\Theta(X, R)$ and $Q(X, 4R)$ in the case of 2 spatial dimensions. *Not to scale.*

the average of u over Θ with respect to the standard space-time measure $d\mu = dt \, d^n x$. In the case of divergence-form operators, we can take $p = 1$.

Note that the exact value of the constant 4 is irrelevant - by changing C and the time offset in Θ , we could decrease it to any value greater than 1. Also observe that $R^{n/(n+1)} \|f\|_{n+1;4R}$ is proportional to

$$\left(\int_{Q(4R)} |f|^{n+1} \right)^{1/(n+1)} R^2,$$

which we can bound by $|f|_{0;4R} R^2$.

We will now show how this inequality can be used to derive a local Hölder estimate with no assumption other than uniform ellipticity. This basic idea can be adapted to give Hölder estimates in more difficult situations - for example, in Chapter 5 we will use it to derive a Hölder estimate for the coefficients of the quasilinear flow (1.0.1). From here on we fix an origin and write $Q(R)$, $\Theta(R)$ for $Q(0, R)$, $\Theta(0, R)$.

Proposition 2.4.21 (Local Hölder Estimate). *Let P satisfy the same assumptions as in Lemma 2.4.20 and assume $u \in W^{2,2}(Q(R))$ is a solution of the equation $Pu = f$ for some bounded f . Then there are constants C and α depending only on n, λ, Λ such that*

$$\operatorname{osc}_{Q(R)} u \leq C \left(\frac{R}{R_0} \right)^\alpha \left(\operatorname{osc}_{Q(R_0)} u + |f|_{0;Q(R_0)} R_0^2 \right) \quad (2.4.1)$$

for all $R \in (0, R_0)$.

Proof. We will assume $p = 1$ for simplicity, but the proof is easily modified for the general case. Let $M_i = \sup_{Q(iR)} u$, $m_i = \inf_{Q(iR)} u$ for $i = 1, 4$, so that $M_4 - u$ and $u - m_4$ are positive supersolutions of $Pu = \pm f$ on $Q(4R)$. Applying the weak Harnack inequality we

obtain

$$\begin{aligned} \int_{\Theta(R)} (M_4 - u) &\leq C \left(M_4 - M_1 + R^{n/(n+1)} \|f\|_{n+1} \right) \\ \int_{\Theta(R)} (u - m_4) &\leq C \left(m_1 - m_4 + R^{n/(n+1)} \|f\|_{n+1} \right) \end{aligned}$$

which we can add together to find

$$\operatorname{osc}_{Q(4R)} u \leq C \left(\operatorname{osc}_{Q(4R)} u - \operatorname{osc}_{Q(R)} u + 2 \|f\|_{n+1} R^{n/(n+1)} \right),$$

which can be rearranged to

$$\operatorname{osc}_{Q(R)} u \leq \left(1 - \frac{1}{C} \right) \operatorname{osc}_{Q(4R)} u + 2 \|f\|_{n+1} R^{n/(n+1)}.$$

Iterating this equality (e.g. use $\omega(R) = \operatorname{osc}_{Q(R)} u$, $\tau = \frac{1}{4}$, $\gamma = 1 - C^{-1}$, $\sigma(R) = 2 \|f\|_{n+1; Q(4R)} R^{n/(n+1)}$ in Lemma 8.23 of [GT83]) and bounding the L^{n+1} average of $|f|$ by its supremum yields the desired estimate. \square

By controlling the coefficient of R^α in (2.4.1), this becomes a uniform Hölder estimate for bounded solutions:

Corollary 2.4.22. *If u is a bounded solution of $Pu = f$ on a closed manifold, then $|u|_{0,\alpha} \leq C(|u|_0 + |f|_0)$ for some C, α depending only on λ, Λ and the manifold geometry.*

Proof. Fix some $R_0 > 0$ smaller than the injectivity radius and work with the R_0 -restricted norm for convenience. Then we have

$$\begin{aligned} [u]_\alpha &= \sup_{d_P(X,Y) < R_0} \frac{|u(X) - u(Y)|}{d_P(X,Y)^\alpha} \\ &= \sup_{R < R_0} \sup_{d_P(X,Y)=R} \frac{|u(X) - u(Y)|}{R^\alpha} \\ &\leq \sup_X \sup_{R < R_0} \frac{1}{R^\alpha} \operatorname{osc}_{Q(X,R)} u. \end{aligned}$$

Substituting in the oscillation estimate (2.4.1), we get

$$[u]_\alpha \leq \sup_X \frac{C}{R_0^\alpha} \left(\operatorname{osc}_{Q(X,R_0)} u + |f|_{0; Q(X,R_0)} R_0^2 \right)$$

and thus in the absolute worst case where u achieves its extreme values in $Q(X, R_0)$

$$|u|_{0,\alpha} \leq |u|_0 + \frac{C}{R_0^\alpha} (2|u|_0 + |f|_0 R_0^2).$$

\square

Sometimes we might be studying a solution satisfying the assumptions of the local estimate Proposition 2.4.21 which is not a priori bounded, in which case we get local Hölder regularity but not a uniform bound on the Hölder norm. To remedy this, we can use a *local maximum principle*, which is an estimate bounding the supremum of the solution on a cylinder in terms of an L^p estimate on a larger cylinder. This result was famously proved for divergence-form equations by Moser using a technique that has become known as *Moser iteration*. We will need it for equations in general form, so we state a simplified (i.e. weakened) version of [Lie96, Theorem 7.21].

Proposition 2.4.23 (Local maximum principle). *If $u \in W^{1,n}(Q(2R))$ satisfies $Pu \geq f$ in $Q(2R)$ for P a uniformly parabolic operator with bounded coefficients, then for any $p \geq 1$ there is a constant C depending only on p, R and the coefficient bounds such that*

$$\sup_{Q(R)} u \leq C \left(\|u^+\|_{p;Q(2R)} + \|f\|_{n+1} \right).$$

Here $u^+(X) = \max\{u(X), 0\}$ is the positive part of u .

In particular, when $Pu = f$ we get $|u|_{0;Q(R)} \lesssim \|u\|_{p;Q(2R)} + \|f\|_{n+1}$; so if u is a solution of a uniformly parabolic equation with bounded source term, *any* L^p bound implies a local supremum bound. This will be very valuable to us in Appendix C.

Chapter 3

Geometric Map Flows

We are interested in defining geometric heat flows of smooth maps $u : M \rightarrow N$; that is, equations of the form $\partial_t u = Eu$ where E is an operator that is elliptic and isometry-invariant in some appropriate senses. We will make a precise definition soon, but the idea is that the flow should become a genuine system of parabolic equations when we fix a coordinate system, which will allow us to apply the standard PDE techniques to studying solutions of our flow. Our main goal is to find flows for which we can solve the Cauchy (or Cauchy-Dirichlet) problem with loose assumptions on the initial data, for which there is a fairly standard program in the quasilinear setting:

1. *Short-time existence:* show that for initial data $u_0 \in C^\infty(M, N)$ (possibly satisfying some assumptions), there exists *some* time interval $[0, \epsilon)$ and a solution $u : M \times [0, \epsilon) \rightarrow N$ of the flow with $u(\cdot, 0) = u_0$. For most flows (including the ones we are considering) this follows in a fairly boring fashion from standard theory; so we will put it off until the end.
2. *First estimates:* show that the solution u constructed in step 1 preserves the assumptions we imposed on u_0 . In our case this will involve obtaining uniform estimates from above and below (in a certain sense) for the gradient.
3. *Hölder gradient estimates:* show that the solution constructed in step 1 has finite $C^{1,\alpha}$ norm with some exponent $\alpha \in (0, 1)$. This will be our biggest hurdle, since there is no general Hölder estimate for systems.
4. *Long-time existence:* using a Schauder bootstrap argument, establish that *all* derivatives of the solution have finite Hölder norm and thus conclude the existence of a smooth limit $\lim_{t \rightarrow \epsilon} u(t)$. Applying short-time existence again then implies the solution exists on some interval $[0, \epsilon + \epsilon')$, so there is no maximal time of existence; that is, the solution exists for all time.
5. *Convergence and well-posedness:* determine the long-time behaviour of the solution, typically using energy estimates or barriers; and establish that the solution operator $u_0 \mapsto u$ is well-defined and continuous.

At some point later we will focus on diffeomorphisms, i.e. $N = M$, but we will still refer to the target as N , since it may be endowed with a different metric. We can also do much of the abstract discussion in full generality without losing any clarity; so for now M and N are just manifolds of the same dimension n . We will sometimes additionally assume that the manifolds are orientable and that u is orientation-preserving.

To properly define our flows, we will need notions of differential operators acting on such maps, which will produce derivatives lying in some bundles over $M \times [0, T)$ or $M \times N$ - the former when we are thinking of a single solution defined on a time interval $[0, T)$, or the latter in the context of jet bundles when we are not fixing a solution (when fixing $x \in M$ does not determine $u(x) \in N$). In the first section of this chapter we will define the various bundles that we will need and develop their natural geometric structure. In following sections we will then use this structure to define parabolic map flows in a coordinate-independent manner, and investigate the constraints on their structure that isometry invariance imposes. Finally, we will derive evolution equations for the derivatives of u , which we will need both to determine which flows preserve the (local) diffeomorphism condition $\det Du \neq 0$, but also to establish the Hölder gradient estimates we need for long-time existence.

3.1 Vector Bundles over Space-time

A solution of a flow of maps $M \rightarrow N$ should be a map $u : M \times I \rightarrow N$ solving some geometric PDE, where $I = [0, T)$ is some (possibly semi-infinite) time interval. We will use the “space-time” $M \times [0, T)$ as our base manifold so we can geometrically describe both temporal and spatial variation, and will often refer to $M \times \mathbb{R}$ instead when the particular time domain is not important.

When using index notation, we will use Latin indices for TM and Greek for TN ; and ∂_α can denote either the usual coordinate field $\partial/\partial y^\alpha \in \Gamma(TN)$ or (more often) the restriction $(\partial/\partial y^\alpha)_u$ considered as a section of u^*TN . This alphabet convention will allow us to use the usual Riemannian notation for Christoffel symbols Γ and curvatures R without any ambiguity as to which manifold we are talking about. Let t and ∂_t denote the natural time coordinate and vector field on $M \times I$ inherited from the product structure.

Definition 3.1.1. The *spatial tangent bundle* is $\mathfrak{S} = \ker dt \subset T(M \times [0, T))$; i.e. the bundle whose fibre at $X = (x, t)$ is $T_x M \simeq T_x M \times \{0\} \subset T(M \times [0, T))$.

This is just a copy of TM for each time, so the metric and connection on M determine corresponding structures on \mathfrak{S} if we prescribe that vector fields independent of time are parallel in the ∂_t direction. Normally when we write Du we mean only the spatial component; i.e. the restriction $\mathfrak{S} \rightarrow u^*TN$. When reasonable to do so we will suppress the time dependence for convenience’s sake - so whenever we write something like $Du \in \Gamma(TM^* \otimes u^*TN)$ we are really referring to the derivative of a spatial slice $u_t := u|_{M \times \{t\}} = u|_{M \times \{t\}}$, which (when concerned only with spatial properties) we will also call u by a slight abuse of notation.

Note that u^*TN can mean two different things depending on whether we are thinking about $u : M \times [0, T) \rightarrow N$ or $u : M \rightarrow N$, but once again u_t^*TN is simply a time slice of the full u^*TN .

The standard constructions from §2.1 then provide us with natural metrics and connections on all tensor products of \mathfrak{S}^* and u^*TN , which allows us to define higher covariant derivatives of u (since $Du \in \Gamma(\mathfrak{S}^* \otimes u^*TN)$); so we have all the machinery we need to write down a parabolic equation for u in the next section. We will often write ∇_t as shorthand for the covariant time derivative ∇_{∂_t} . On the topic of the time derivative, we need to be careful with how it acts on u^*TN : while ∂_α seems like it should be constant in time when viewed as a vector field on N , as a section of u^*TN we have $\nabla_t \partial_\alpha = \partial_t u^\beta \Gamma_{\beta\alpha}^\gamma \partial_\gamma$, which is non-zero in general. We can make use of this formula to prove the following elementary but useful fact:

Proposition 3.1.2. $\nabla_t Du = \nabla \partial_t u$.

Proof. On the left we have

$$\nabla_t (\partial_i u^\alpha dx^i \otimes \partial_\alpha) = \partial_t \partial_i u^\alpha dx^i \otimes \partial_\alpha + \partial_i u^\alpha dx^i \otimes \nabla_t \partial_\alpha = \partial_t \partial_i u^\alpha dx^i \otimes \partial_\alpha + \partial_i u^\alpha \partial_t u^\beta \Gamma_{\beta\alpha}^\gamma dx^i \otimes \partial_\gamma$$

while on the right we have

$$\nabla (\partial_t u^\alpha \partial_\alpha) = \partial_i \partial_t u^\alpha dx^i \otimes \partial_\alpha + \partial_t u^\alpha \nabla \partial_\alpha = \partial_i \partial_t u^\alpha dx^i \otimes \partial_\alpha + \partial_t u^\alpha \partial_i u^\beta \Gamma_{\beta\alpha}^\gamma dx^i \otimes \partial_\gamma;$$

so the symmetry of partial derivatives and the fact that the Levi-Civita connection of N is torsion-free are all we need. \square

Writing this as $\nabla_t \nabla_i u = \nabla_i \nabla_t u$, we see that this is really just a manifestation of the fact that the first two covariant derivatives commute for a manifold-valued map; cf. [O’N83, Chap 4, Prop 44].

In order to derive evolution equations for Du (and higher derivatives), we will need to commute covariant derivatives; so we will also need to know the curvature of the natural connection induced on the tensor bundle $\mathfrak{S}^* \otimes u^*TN$. We can get a formula for this by applying all three parts of Proposition 2.1.11:

Proposition 3.1.3. *The curvature of the natural connection on $\mathfrak{S}^* \otimes u^*TN$ is given by*

$$R_{ijk}^{\alpha l} = -R_{ij}^{\quad l}{}_k \delta_\beta^\alpha + \nabla_i u^\mu \nabla_j u^\nu (R_{\mu\nu}^{\quad \alpha}{}_\beta \circ u) \delta_k^l \quad (3.1.1)$$

where the R s appearing on the RHS are the Riemannian curvatures of $M \times \mathbb{R}$ and N respectively.

We will often leave the composition with u implicit and write this more compactly as $-R_{ij}^{\quad l}{}_k \delta_\beta^\alpha + u_i^\mu u_j^\nu R_{\mu\nu}^{\quad \alpha}{}_\beta \delta_k^l$ - the point at which the curvature of N must be evaluated is determined by the fact that it is contracted with Du .

3.2 Harmonic Map Heat Flow

The most obvious heat-like flow we can write down for a map u is the direct analog of the heat equation:

$$\partial_t u = \Delta u \quad (3.2.1)$$

where Δu is the (*harmonic*) *map Laplacian* $\text{tr}_g \nabla^2 u$, often known instead as the *tension field* of u . Here $\nabla^2 u = \nabla (Du) \in \Gamma(\mathfrak{S}^* \otimes \mathfrak{S}^* \otimes u^*TN)$ is the second covariant derivative of u in the spatial directions. In abstract index notation the tension field is simply $(\Delta u)^\alpha = g^{ij} \nabla_i \nabla_j u^\alpha$, while the second derivative has the local coordinate expression

$$\nabla_i \nabla_j u^\alpha = \partial_i \partial_j u^\alpha - \Gamma_{ij}^k \partial_k u^\alpha + \partial_i u^\gamma \Gamma_{\gamma\beta}^\alpha \partial_j u^\beta \quad (3.2.2)$$

for $\Gamma_{ij}^k, \Gamma_{\gamma\beta}^\alpha$ the Christoffel symbols of M, N respectively. As in the case of the scalar heat equation, this is the L^2 gradient descent flow for the Dirichlet energy $\frac{1}{2} \int |Du|^2$. Equation (3.2.1) is called the *harmonic map heat flow* (henceforth HMHF) for u , which produces harmonic limits in a quite general setting:

Theorem. [ES64, Har67] *If $u_0 : M \rightarrow N$ is a map between compact Riemannian manifolds and N has nonpositive sectional curvature, then the HMHF Cauchy problem*

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u \\ u(x, 0) = u_0(x) \end{cases}$$

has a unique smooth solution $u : M \times [0, \infty) \rightarrow N$. If u_0 is not null-homotopic then u converges smoothly to a harmonic map $u_\infty : M \rightarrow N$ as $t \rightarrow \infty$.

However, the HMHF is not well-suited to studying diffeomorphisms: if $u_0 : M \rightarrow N$ is a diffeomorphism, it's possible that $u(\cdot, t)$ is not a diffeomorphism at later times $t > 0$. We will construct an example illustrating this on the flat torus, with the help of a simple well-posedness result:

Lemma 3.2.1. *The harmonic map heat flow into (a quotient of) \mathbb{R}^n is well-posed in C^k : in particular, if $u, v : M \rightarrow \mathbb{R}^n$ are solutions of HMHF, then $|u(\cdot, t) - v(\cdot, t)|_k \leq |u(\cdot, 0) - v(\cdot, 0)|_k$ for all times $t > 0$.*

Proof. Let α be a multi-index of size $|\alpha| \leq k$ and let $w = \partial^\alpha u - \partial^\alpha v$, so that it suffices to show $\|w(\cdot, t)\|_\infty \leq \|w(\cdot, 0)\|_\infty$. Since the components of u and v solve the heat equation, the components of w do too, and thus the parabolic maximum principle yields the desired inequality. \square

Now to the promised example of a diffeomorphism degenerating:

Example 3.2.2. Let \mathbb{T}^2 be the flat torus constructed by gluing edges of the square $[-\pi, \pi]^2$, $B = B_R(0) \subset \mathbb{T}^2$ with $R < 1$ to be chosen later, and for $\epsilon \in (0, 1)$ define a map $u^\epsilon : B \rightarrow \mathbb{T}^2$

by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -\frac{1}{2}(1+x)y^2 + \epsilon x + x^5 \\ (1+x)y \end{pmatrix}.$$

One can check that as a map $B \rightarrow \mathbb{R}^2$, this is an injective immersion so long as $(x, y) \in B_1$; and by choosing R small enough we can make it an embedding $B \rightarrow \mathbb{T}^2$. This map is carefully chosen near zero to lie ϵ -close to the boundary of the set of diffeomorphisms with time evolution pointing out of this set - we will see a more systematic approach to understanding (and preventing!) this in Chapter 4. The key property is that the Jacobian determinant $\rho = \det Du$ is positive everywhere with minimum ϵ at the origin, but under the harmonic map heat flow will have negative time derivative. The plan is to extend this map to a diffeomorphism of \mathbb{T}^2 and then let $\epsilon \rightarrow 0$.

To do this extension, we first produce an isotopy from u^ϵ to the standard embedding $\iota : B \rightarrow \mathbb{T}^2$. Start by defining $u_t^\epsilon(x) = \frac{1}{1-t}u^\epsilon((1-t)x)$ for $t \in [0, 1)$, so that $u_0^\epsilon = u^\epsilon$. Since $\lim_{t \rightarrow 1} u_t^\epsilon$ is by definition

$$L^\epsilon := Du^\epsilon(0) = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix},$$

u_t^ϵ constitutes an isotopy from u^ϵ to the linear embedding $L^\epsilon : B \rightarrow \mathbb{T}^2$; and the generating vector field $\partial_t u_t^\epsilon$ has uniform bounds on all derivatives independent of ϵ . Now define u_t^ϵ on $(1, 2]$ by $u_t^\epsilon(x) = (t-1)\text{id} + (2-t)L^\epsilon$, so that $u^\epsilon : [0, 2] \times B \rightarrow \mathbb{T}^2$ is a piecewise-smooth isotopy from u^ϵ to the identity. Once again, the generating vector field is uniformly bounded in any C^k independent of ϵ . To finish the extension, let ψ be a smooth cutoff function with compact support inside B and equal to 1 on some neighbourhood U of the origin, define the time-dependent vector field $X^\epsilon(x, t) = -\psi(x)\partial_t u_{2-2t}^\epsilon(x)$ on all of \mathbb{T}^2 and let $\tilde{u}_t^\epsilon : [0, 1] \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be the isotopy generated by X^ϵ . Then the endpoint $\tilde{u}^\epsilon := \tilde{u}_1^\epsilon$ is a diffeomorphism of \mathbb{T}^2 agreeing with u^ϵ on U , so its 3-jet at the origin agrees with that of u^ϵ ; and $|\tilde{u}^\epsilon|_4$ is bounded independent of ϵ . Applying the Arzela-Ascoli theorem we can thus extract a sequence $\epsilon_j \rightarrow 0$ such that the maps \tilde{u}^{ϵ_j} converge to a limit \tilde{u} in C^3 .

Now let $U^j : \mathbb{T}^2 \times [0, \infty) \rightarrow \mathbb{T}^2$ be the solution of HMHF with initial condition \tilde{u}^{ϵ_j} , and U the solution with initial condition \tilde{u} . Since C^3 convergence implies pointwise convergence of 3-jets, at $(x, y, t) = (0, 0, 0)$ we can compute

$$DU = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \partial_t DU = \Delta DU = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

so for some $t_0 > 0$ we know that $\det DU_{t_0}(0) < 0$. Since $\tilde{u}^{\epsilon_j} \rightarrow \tilde{u}$ in C^1 and $\tilde{u}^{\epsilon_j} \mapsto U^j$ is continuous in C^1 (by Lemma 3.2.1), this means that there is some j such that $\det DU_{t_0}^j(0) < 0$. But we know $\det DU_0^j(0) > 0$, so by continuity $\det DU_t^j(0)$ must be zero at some time. That is, U^j is a solution to the HMHF with diffeomorphic initial data that fails to be a diffeomorphism at a later time.

Note that the failure here is quite generic: while we used \mathbb{T}^2 for convenience, the local nature of the example suggests we could construct a similar family of diffeomorphisms on any manifold. This failure of the HMHF provides a large part of the motivation for this thesis - we want to find a modification of the HMHF that stays inside $\text{Diff}(M)$ whilst retaining the existence and convergence results. We will do this by investigating the properties of more general parabolic evolution equations, which motivates the next section.

3.3 Quasilinear Map Flows

We can use the geometric structure on \mathfrak{S} and u^*TN discussed earlier to extend the usual notion of a parabolic PDE to $u : M \times [0, T) \rightarrow N$. We might think to define a “linear” parabolic map flow as an equation of the form

$$\partial_t u^\alpha(X) = a^{ij}(x) \nabla_i \nabla_j u^\alpha(X) + b^i(x) \nabla_i u^\alpha(X) \quad (3.3.1)$$

where $a \in \Gamma(TM \otimes TM)$ satisfies the ellipticity condition $a^{ij}(x) v_i v_j \geq \lambda |v|^2$ for every $v \in \mathfrak{S}^*$ and $b \in \Gamma(TM)$. (Note that despite the innocuous appearance in index notation, it doesn’t make sense to add a $c(x)u^\alpha(X)$ term: u^α are the coordinates of a *point* in N , while $\partial_t u^\alpha$ needs to be a vector tangent to N .) In particular if we choose $a = g^{-1}$, $b = 0$ we get HMHF. However, calling this equation linear is a misnomer, since the fact that the target is only a manifold means there is no natural vector space structure on the solutions. Even if we fix a coordinate system, the leading term becomes

$$a^{ij} \left(\partial_i \partial_j u^\alpha - \Gamma_{ij}^k \partial_k u^\alpha + \Gamma_{\beta\gamma}^\alpha \partial_i u^\beta \partial_j u^\gamma \right),$$

the last component of which is clearly not linear in u unless N is flat and y^α are affine coordinates. Still, one might hope that the equations of this form might be easier to get estimates for than more general ones. Unfortunately, while this is the case, we will soon see that in order to be isometry-invariant (a necessary condition if we want our flow to make sense in a general setting), a “linear” map flow is necessarily a multiple of HMHF; so there’s no new behaviour to be found here. Thus we move on to quasilinear equations, where we expect to find much more varied behaviour.

Definition 3.3.1. A *quasilinear map flow* is an equation of the form

$$\partial_t u^\alpha(X) = a^{ij\alpha}_\beta(j_x^1 u_t) \nabla_i \nabla_j u^\beta(X) + b^\alpha(j_x^1 u_t) \quad (3.3.2)$$

where $a : J^1(M, N) \rightarrow \text{Sym}^2 TM \otimes \text{End}(TN)$, $b : J^1(M, N) \rightarrow TN$ preserve fibres over $M \times N, N$ respectively and a satisfies the Legendre-Hadamard condition: there must exist a positive constant λ such that $a^{ij\alpha}_\beta(\xi) v_i v_j z_\alpha z_\gamma g_N^{\beta\gamma} \geq \lambda |v|^2 |z|^2$.

To be concrete, a and b take a 1-jet (i.e. a point $x \in M$, a point $z \in N$ and a linear map $\xi : T_x M \rightarrow T_z N$), with a producing an element of $\text{Sym}^2 T_x M \otimes \text{End}(T_z N)$ and b an

element of $T_z N$. The fibre-preserving condition on b makes sense by considering $J^1(M, N)$ as a bundle over N by composing the usual bundle map with the product projection: for example, the requirement that $b(j_x^1 u) \in T_{u(x)} N$ is equivalent to the diagram

$$\begin{array}{ccc} J^1(M, N) & \xrightarrow{b} & TN \\ \downarrow & & \downarrow \\ M \times N & \longrightarrow & N \end{array}$$

commuting. These conditions are required so that the pairing $a^{ij\alpha} \nabla_i \nabla_j u^\beta$ and the sum $a^{ij\alpha} \nabla_i \nabla_j u^\beta + b^\alpha$ make sense: we need $a^{ij\alpha}$ and $\nabla_i \nabla_j u$ to have indices i, j from the same tangent space $T_x M$, and $\partial_t u^\alpha$, $\nabla^2 u^\alpha$, $a^{ij\alpha}$ and b^α must all have Greek indices in the same tangent space $T_{u(x)} N$.

In practice we will nearly always be considering diffusive coefficients of the form $a^{ij\alpha} = a^{ij} \delta_\beta^\alpha$ - this is mostly because we can then define the related operator $P = \partial_t - a^{ij} \nabla_i \nabla_j$ for objects other than maps. In the next chapter we will see that the close relationship between flows of this form and the scalar differential operator P allows us to attain maximum principles for scalar geometric invariants.

We will sometimes write $a^{ij}(x, u(X), Du(X))$ or simply $a^{ij}(Du)$ instead of $a^{ij}(j_x^1 u_t)$ - the first to highlight the similarity to the familiar scalar case and the second because $Du(X) : T_x M \rightarrow T_{u(X)} N$ determines both source and target points, so knowing the differential is really the same thing as knowing the 1-jet. The reason we defined equation (3.3.2) in terms of jets (instead of e.g. $Du \in T^* M \otimes u^* TN$) is that it gives a natural domain for the coefficients a^{ij} , b^α independent of a candidate solution u . This will allow us to investigate the effect of isometry invariance in the next section.

3.4 Invariant Map Flows

3.4.1 Isometry Invariance

In principle, if we had particular spaces in mind for domain and target we could choose distinguished coordinate systems (or embeddings, etc) and use these to define our flow. As discussed above, however, we would like to keep things invariant so that they apply in as general a setting as possible. It also tends to be the case that geometrically defined equations are nicer to work with - in almost all the famous examples of geometric flows, various estimates are obtained first for geometrically defined curvatures or energy densities, which tend to interact very nicely with isometry-invariant evolution equations.

Thus we come to our next question: what do the defining equations of geometrically defined quasilinear flows look like? The most naive definition is in terms of Riemannian isometries:

Definition 3.4.1. An equation for maps $u : M \times [0, T) \rightarrow N$ is *locally isometry invariant*

if whenever $t \mapsto u_t$ is a local solution and ϕ, ψ are local isometries of M, N respectively with appropriate domains/targets, $t \mapsto \psi \circ u_t \circ \phi$ is a local solution.

For particular choices of M, N this might not mean much - the less symmetric the spaces, the less restrictive this condition. One issue, though, is that we'd have no way of transplanting such a flow from one manifold to another - the background anisotropy in effect would become a structure on the manifold upon which the definition of the flow relies. If we want to be able to define a flow that makes sense for any choice of metrics (like HMHF), we need to require isometry invariance for the metrics with the most local isometries, which are those of constant curvature. Intuitively this means there is no background anisotropy to induce preferred directions for the flow.

Remark. Actually, this is not quite true as written - for example, the flow

$$\partial_t u^\alpha = \left(g^{ij} + (\text{Rc}^M)^{ij} \right) \nabla_i \nabla_j u^\alpha \quad (3.4.1)$$

makes sense and is isometry-invariant on any Riemannian manifold; so we could allow preferred directions induced by background geometry. Thus the most general definition of a “natural flow” would allow the coefficients to depend on (rotationally invariant) functions of the curvature; cf. the notion of *natural tensors* developed in [Eps75]. However, we will be avoiding these terms: they do not appear in the constant-curvature case (since all curvature terms there can be expressed in terms of the curvature constant and the metric), and the analysis in this case is already quite difficult as we will see. In general the flow (3.4.1) is not even parabolic - we need the curvature bound $\text{Rc}^M > -g$ to make the coefficients positive-definite.

In the constant-curvature setting, isometry invariance can be stated in terms of linear isometries between tangent spaces:

Proposition 3.4.2. *Let M, N have constant curvature and $\partial_t u^\alpha = a^{ij}(Du) \nabla_i \nabla_j u^\alpha + b^\alpha(Du)$ be a quasilinear flow of maps $M \rightarrow N$. If the flow is locally isometry invariant and for every jet $\zeta \in J^2(M, N)$ there is a solution $u : M \times [0, \epsilon) \rightarrow N$ contacting ζ (i.e. $u_0 \in \zeta$), then the flow coefficients satisfy:*

- Left-invariance: $a^{ij}(\xi) = a^{ij}(\Psi\xi)$ for any linear isometry $\Psi : T_z N \rightarrow T_w N$ and any linear $\xi : T_x M \rightarrow T_z N$,
- Right-equivariance: $a^{ij}(\xi) = a^{kl}(\xi\Phi) \Phi_k^i \Phi_l^j$ for any linear isometry $\Phi : T_y M \rightarrow T_x M$ and any linear $\xi : T_x M \rightarrow T_z N$,
- No advection/reaction: $b = 0$.

Proof. The condition on the existence of solutions means that we can assume $\xi = Du(X)$ for some solution u and point X . Since M, N have constant curvature, any linear isometry

$\Psi : T_{u(X)}N \rightarrow T_qN$ is the derivative of the local isometry $\psi = \exp_q \circ \Psi \circ \log_{u(X)}$ defined on some ball centred on $u(X)$. For $\psi \circ u$ to be a solution we need

$$\partial_t (\psi \circ u)^\alpha = a^{ij} (D(\psi \circ u)) \nabla_i \nabla_j (\psi \circ u)^\alpha + b^\alpha (D(\psi \circ u)).$$

Working in some geodesic normal coordinates about $x, u(X)$ so that $\nabla_i \nabla_j u(X) = \partial_i \partial_j u$ and ψ has linear coordinate representative Ψ , at X this becomes

$$\Psi_\beta^\alpha \partial_t u^\beta = \Psi_\beta^\alpha a^{ij} (\Psi Du) \nabla_i \nabla_j u^\beta + b^\alpha (\Psi Du).$$

Substituting $\partial_t u^\beta = a^{ij} (Du) \nabla_i \nabla_j u^\beta$, we get that the invariance of the solution u under the isometry ψ implies

$$\Psi_\beta^\alpha a^{ij} (Du) \nabla_i \nabla_j u^\beta + \Psi_\beta^\alpha b^\beta (Du) = \Psi_\beta^\alpha a^{ij} (\Psi Du) \nabla_i \nabla_j u^\beta + b^\alpha (\Psi Du).$$

Since the existence condition implies $\nabla_i \nabla_j u^\alpha$ can take on any value, we conclude that $a^{ij} (\Psi Du) = a^{ij} (Du)$ and $b^\alpha (\Psi Du) = \Psi_\beta^\alpha b^\beta (Du)$. Following the same path with $u \circ \phi$ where $\phi = \exp_x \circ \Phi \circ \log_y$ we get

$$a^{ij} (Du) \nabla_i \nabla_j u^\alpha + b^\alpha (Du) = a^{ij} (Du \Phi) \nabla_k \nabla_l u^\alpha \Phi_i^k \Phi_j^l + b^\alpha (Du \Phi)$$

and thus conclude $a^{kl} (Du) = a^{ij} (Du \Phi) \Phi_i^k \Phi_j^l$ and $b^\alpha (Du \Phi) = b^\alpha (Du)$. To prove the third point, note that $Du(-\text{id}_{T_x M}) = -\text{id}_{T_{u(x)} N} Du$, so applying both transformation laws for b that we obtained above we get $b^\alpha (Du) = -b^\alpha (Du)$. \square

This version of invariance should help us find “portable” flows - it is now a quite restrictive condition even when M, N have few isometries. We will see in a second that it in fact implies an explicit geometric form of the flow equation. Before doing this in full generality, we first show how it works in the case of scalar PDE on manifolds (i.e. $N = \mathbb{R}$), where the resulting form of the equation is well known (see e.g. [AC13]).

Example 3.4.3. Let $N = \mathbb{R}$ and consider a quasilinear PDE

$$\partial_t u = a^{ij} (x, u(x), \nabla u(x)) \nabla_i \nabla_j u$$

that is invariant in the sense of Proposition 3.4.2. The first thing we notice is that the left-invariance and the natural identification of all tangent spaces of \mathbb{R} implies that a^{ij} is independent of u . The equivariance is then

$$a^{kl} (x, du(x)) = a^{ij} (y, \Phi^* du(x)) \Phi_i^k \Phi_j^l$$

for every linear isometry $\Phi : T_y M \rightarrow T_x M$. First let’s investigate the pointwise implications of this. If $\Phi \in O(T_x M)$ fixes $du(x)$ then we get $a^{kl} (x, du(x)) = a^{ij} (x, du(x)) \Phi_i^k \Phi_j^l$, so $a^{ij} (x, du(x))$ is an $O(m-1)$ -invariant quadratic form when restricted to the hyperplane

$du(x)^\perp \subset T_x M^*$ and is thus a multiple of g^{-1} there. If $\xi \perp du(x)$ then there is some $\Phi \in O(T_x M)$ fixing $du(x)$ such that $\Phi^* \xi = -\xi$, so we get

$$a^{kl}(x, du(x)) du_k(x) \xi_l = -a^{ij}(x, du(x)) du_i(x) \xi_j$$

and thus $a^{ij}(x, du(x))$ vanishes on $\text{span } du(x) \times du(x)^\perp$. Putting these facts together we can conclude that

$$a^{ij}(x, du(x)) = \alpha(x, du(x)) \nabla^i u(x) \nabla^j u(x) + \beta(x, du(x)) \left(g^{ij}(x) - \frac{\nabla^i u(x) \nabla^j u(x)}{|du(x)|^2} \right)$$

for some functions α, β . Applying the equivariance to this expression we get

$$\begin{aligned} \alpha(x, du) \nabla^k u \nabla^l u + \beta(x, du) \left(g^{kl} - \frac{\nabla^k u \nabla^l u}{|du|^2} \right) \\ = \left(\alpha(y, \Phi^* du) \nabla^i u \nabla^j u + \beta(y, \Phi^* du) \left(g^{ij} - \frac{\nabla^i u \nabla^j u}{|du|^2} \right) \right) \Phi_i^k \Phi_j^l \end{aligned}$$

In particular if Φ is a linear isometry such that $du(y) = \Phi^*(du(x))$ then we have $\Phi_i^k \nabla^i u(y) = \nabla^k u(x)$; so we get

$$\begin{aligned} \alpha(x, du(x)) \nabla^k u \nabla^l u + \beta(x, du(x)) \left(g^{kl} - \frac{\nabla^k u \nabla^l u}{|du|^2} \right) \\ = \alpha(y, du(y)) \nabla^k u \nabla^l u + \beta(y, du(y)) \left(g^{kl} - \frac{\nabla^k u \nabla^l u}{|du|^2} \right). \end{aligned}$$

This implies $\alpha(x, du(x)) = \alpha(y, du(y))$ and $\beta(x, du(x)) = \beta(y, du(y))$ whenever there is such a Φ , which is exactly when $du(x)$ and $du(y)$ have the same norm. Thus we can in fact write

$$a^{ij} = \alpha(|du|) \nabla^i u \nabla^j u + \beta(|du|) \left(g^{ij} - \frac{\nabla^i u \nabla^j u}{|du|^2} \right).$$

After a little thought, this is exactly what we expect a geometrically defined second-order quasilinear PDE to look like - the only first-order geometric scalar quantity is $|du|$, and the only geometrically privileged direction is $\nabla u / |\nabla u|$. Thus we can have an isotropic component βg^{-1} and a directional component $(\alpha - \beta |du|^{-2}) \nabla u \otimes \nabla u$ with the coefficients α, β depending only on $|du|$.

We will now see how this explicit characterization of isometry invariance can be extended to general N -valued flows. In order to propose a candidate form of the coefficients, we need to think about the geometric invariants of the derivative Du . In the scalar case the derivative could be viewed as a vector, so the only invariant scalar was $|du|$ and the only invariant direction was ∇u . In order to discuss the corresponding geometric invariants of Du in the general case, we will need a brief detour in to the land of matrix decompositions.

3.4.2 The Singular Value and Polar Decompositions

The geometric invariants of a linear map between inner product spaces are described by the *singular value decomposition* (henceforth *SVD*). We first recall the classical SVD for square matrices:

Proposition 3.4.4. *For any matrix $\xi \in \mathbb{R}^{n \times n}$, there exist orthonormal matrices V, E and a non-negative diagonal matrix U such that $\xi = VUE^T$. The matrix U is unique up to permutations, and if the entries of U are distinct then the corresponding V, E are unique up to reflections.*

Proof. Since $\xi^T \xi$ is a non-negative definite symmetric matrix, it can be diagonalized as

$$\xi^T \xi = EU^2E^T$$

where $E \in O(n)$ and U is non-negative diagonal. Defining $V = \xi EU^{-1}$ we see that $\xi \xi^T = VU^2V^T$ and

$$VV^T = \xi EU^{-2}E^T \xi^T = \xi (\xi^T \xi)^{-1} \xi^T = I,$$

so V is also orthogonal. We can now check that

$$VUE^T = \xi EU^{-1}UE^T = \xi,$$

so we have the desired decomposition. Now assume we have *any* such decomposition $\xi = VUE^T$, and note that $\xi^T \xi = EU^2E^T$ must hold; so the entries of U^2 are necessarily the eigenvalues of $\xi^T \xi$, and are thus uniquely determined (up to order) by ξ ; so with the requirement of non-negativity we see the entries of U are also uniquely determined up to order. Since we have $\xi^T \xi = EU^2E^T$ and $\xi \xi^T = VU^2V^T$, U having distinct entries is equivalent to these matrices having distinct eigenvalues; so with the constraint of being unit vectors, the column vectors of E, V (i.e. eigenvectors of these derived matrices) are uniquely determined up to reflection. \square

We call the entries of U the *singular values* of ξ , and the corresponding columns of E and V the *left-* and *right-singular vectors* of ξ .

Since we will be applying this decomposition to varying quantities, we will need some additional facts on the regularity of the singular values and vectors which are easily obtained from the corresponding results for eigensystems. Continuity holds in general:

Proposition 3.4.5. *The map $GL(n, \mathbb{R}) \rightarrow \mathbb{R}$ that sends a matrix to its k^{th} smallest singular value is continuous.*

Away from *exceptional points* where two or more singular values coincide, we have full smoothness:

Proposition 3.4.6. *Let $\mathcal{U} \subset GL(n, \mathbb{R})$ denote the (open) set of matrices with positive, distinct singular values. For any point $\zeta \in \mathcal{U}$, we can choose a neighbourhood \mathcal{V} of ζ in \mathcal{U}*

and smooth functions $U : \mathcal{V} \rightarrow \text{Diag}(n)$, $E, V : \mathcal{V} \rightarrow O(n)$ such that $V(\xi)U(\xi)E(\xi)^T = \xi$ for all $\xi \in \mathcal{V}$.

Near exceptional points, things can get uglier; but we at least have directional/partial differentiability:

Proposition 3.4.7. *For any curve $\zeta : (-\epsilon, \epsilon) \rightarrow GL(n, \mathbb{R})$, there exists a differentiable function $U : I \rightarrow \text{Diag}(n)$ and functions $E, V : I \rightarrow O(n)$ such that $V(t)U(t)E(t)^T = \zeta(t)$.*

Proofs. See e.g. [Kat82] for the corresponding results for eigenvalues/vectors of diagonalizable matrices - we have full smoothness away from the exceptional set and directional differentiability of the eigenvalues everywhere. Since the maps $\xi \mapsto \xi^T \xi$ and $\xi \mapsto \xi \xi^T$ are smooth, the regularity of the eigenvalues and eigenvectors of $\xi^T \xi$ and $\xi \xi^T$ gives us regularity of E, V and U^2 ; and the fact that the singular values are positive in $GL(n, \mathbb{R})$ means that taking the square root to get U preserves this regularity. \square

To apply the SVD to our problem we can simply view the differential $Du(x) : T_x M \rightarrow T_{u(x)} N$ as a matrix after fixing arbitrary orthonormal frames on the domain and target, and then apply the above results:

Proposition 3.4.8. *Let M, N be Riemannian manifolds of the same dimension. For any $\xi \in L(T_p M, T_q N) \subset J^1(M, N)$, there is an orthonormal basis e_i for $T_p M$, an orthonormal basis v_i for $T_q N$ and a collection of scalars σ_i such that*

$$\xi(e_i) = \sigma_i v_i, \quad i \in \{1, \dots, m\}.$$

Furthermore:

- The singular values σ_i are unique up to ordering;
- The singular values can be locally chosen to be continuous and directionally differentiable functions on $J^1(M, N)$;
- On the open subset of $J^1(M, N)$ where the singular values σ_i are all distinct, the singular values are smooth and the corresponding left-singular vectors v_i and right-singular vectors e_i are unique and smooth.

Proof of Prop 3.4.8. Since all the desired conclusions are local, fix neighbourhoods \mathcal{U}, \mathcal{V} with smooth orthonormal frames α, β around $x_0 \in M, y_0 \in N$. For $(x, y) \in \mathcal{U} \times \mathcal{V}$ and $\xi \in J_x^1(M, N)_y$, define $\Xi(\xi)$ by $\Xi_j^i = \langle \beta_i, \xi(\alpha_j) \rangle$ and note that Ξ is a smooth map $J^1(\mathcal{U}, \mathcal{V}) \rightarrow \mathbb{R}^{n \times n}$. Then

$$\begin{aligned} e_i(\xi) &= E_i^j(\Xi(\xi))\alpha_j \\ \sigma_i(\xi) &= U_{ii}(\Xi(\xi)) \\ v_i(\xi) &= V_i^j(\Xi(\xi))\beta_j \end{aligned}$$

(where E, U, V are the SVD components from Proposition 3.4.4) satisfy the desired equations, and the uniqueness and regularity follows directly from the corresponding results in Propositions 3.4.5 - 3.4.7. \square

If $u : M \rightarrow N$ is a slice of our flow map, then composing the above maps with the prolongation $j^1 u : M \rightarrow J^1(M, N)$ will produce geometric quantities on M with which we can analyse the flow: consider $e = (e_1 \cdots e_n)$, $U = \text{diag}(\sigma_1, \dots, \sigma_n)$ and $v = (v_1 \cdots v_n)$ as (local) sections of the bundles $F_O(TM)$, $M \times GL(n, \mathbb{R})$, $F_O(u^*TN)$ respectively, so that $Du = vUe^{-1}$ is the natural composition

$$TM \xrightarrow{e^{-1}} \mathbb{R}^n \xrightarrow{U} \mathbb{R}^n \xrightarrow{v} u^*TN.$$

Unfortunately, due to the lack of regularity of the SVD, this decomposition is only smooth when Du has no exceptional points on the domain of interest, so while it will be useful for describing the geometric form of invariant flows and choosing coordinates adapted to the flow, it will not help us in the harder analysis we will do later, particularly in Chapter 5. Thus we will sometimes use the *polar decomposition* instead, which is a related decomposition of an arbitrary matrix in to symmetric and orthogonal parts. The *left* and *right polar decompositions* of ξ are defined by

$$\xi = L\Theta \quad \xi = \Theta R$$

respectively, where Θ is orthogonal and L, R are symmetric positive semidefinite. When ξ is invertible, the symmetric factors are positive-definite and the decomposition is unique. The factors L, R, Θ can be given in terms of the singular value decomposition $\xi = VUE^T$ as

$$L = VUV^T \quad R = EUE^T \quad \Theta = VE^T.$$

The intuition here is that the polar decomposition factors an arbitrary linear transformation in to a rotation and some stretches in various directions, while the SVD takes the stretches to be along the standard axes, at the cost of an extra rotation. In terms of the singular values and vectors, we can express these relationships as $\Theta e_i = v_i$, $Re_i = \sigma_i e_i$ and $Lv_i = \sigma_i v_i$.

While the polar decomposition does not explicitly extract the geometric scalars from ξ , it has the advantage of global regularity:

Proposition 3.4.9. *The polar decomposition maps $L, R : GL(n, \mathbb{R}) \rightarrow \text{Sym}_{n \times n}^+$ and $\Theta : GL(n, \mathbb{R}) \rightarrow O(n, \mathbb{R})$ are smooth.*

Proof. Note from the proof of Proposition 3.4.4 that we have $L(\xi) = \sqrt{\xi \xi^T}$ and $R(\xi) = \sqrt{\xi^T \xi}$, so the smoothness of the matrix square root $\sqrt{\cdot} : \text{Sym}_{n \times n}^+ \rightarrow \text{Sym}_{n \times n}^+$ immediately tells us that L, R are smooth. Since matrix inversion on $\text{Sym}_{n \times n}^+$ is also smooth, the equation $\Theta = L^{-1}\xi$ along with smoothness of L implies smoothness of Θ . \square

In a very similar fashion to Proposition 3.4.8, we can transfer the right polar decomposition to manifolds using local orthonormal frames along with the natural behaviour of the polar decomposition under orthogonal transformations:

Proposition 3.4.10. *Let M, N be Riemannian manifolds of the same dimension, $J_{\text{inv}}^1(M, N) \subset J^1(M, N)$ the smooth subbundle of invertible jets and π_M, π_N the natural projections from $M \times N$. Then there is a smooth bundle map*

$$(S, \Theta) : J_{\text{inv}}^1(M, N) \rightarrow GL(\pi_M^* TM) \times J_{\text{inv}}^1(M, N)$$

satisfying $\Theta(\xi) \circ S(\xi) = \xi$ such that $S(\xi)$ is self-adjoint and $\Theta(\xi)$ is isometric for any $\xi \in J_{\text{inv}}^1(M, N)$.

By composing this with the prolongation of a given immersion $u : M \rightarrow N$, we obtain a smooth self-adjoint $S_u \in \Gamma(GL(\mathfrak{S}))$ and a smooth isometric $\Theta_u \in \Gamma(SO(\mathfrak{S}, u^*TN))$ such that $\Theta_u \circ S_u = Du$. (When E, F are oriented vector bundles equipped with fibre metrics, we use the notation $SO(E, F)$ to denote the subbundle of $\text{Hom}(E, F)$ consisting of positively-oriented linear isometries.) If $u : M \times [0, T] \rightarrow N$ varies smoothly in time then the corresponding time-dependent sections Θ, S will also.

Clearly we could also have defined a corresponding left polar decomposition in a similar way; but it simplifies things to work with the self-adjoint part acting on TM rather than u^*TN . When the map u is clear from context, we will call $\Theta = \Theta_u$ the *rotational component* of Du , and similarly $S = S_u$ the *stretch component*.

3.4.3 Geometric Form of Invariant Flows

Now that we have developed the SVD for maps between Riemannian manifolds, we can deliver the promised geometric form of the flow equation:

Proposition 3.4.11. *If $u_t^\alpha = a^{ij}(Du) u_{ij}^\alpha$ is a locally isometry invariant flow, then*

$$a^{ij}(Du) = \sum_k \tilde{a}^{kl}(\sigma_1, \dots, \sigma_n) e_k^i e_l^j \quad (3.4.2)$$

where $\tilde{a}^{kl}(\dots)$ is the diagonal matrix with components

$$\tilde{a}^{kk}(\sigma_1, \dots, \sigma_n) = F(\sigma_k; \sigma_1, \dots, \widehat{\sigma_k}, \dots, \sigma_n)$$

for some positive function $F : \mathbb{R} \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ that is symmetric in the last $n-1$ arguments. (Here $\widehat{\sigma_k}$ denotes omission.)

Proof. Fix points $x_0 \in M, y_0 \in N$ and orthonormal bases α for $T_{x_0}M$ and β for $T_{y_0}N$. Let $Du \in L(T_{x_0}M, T_{y_0}N)$ be arbitrary with SVD $Du(e_i) = \sigma_i v_i$ and define $\xi \in L(T_{x_0}M, T_{y_0}M)$ by $\xi(\alpha_i) = \sigma_i \beta_i$. Then the linear isometries defined by $\Phi(\alpha_i) = e_i, \Psi(\beta_i) = v_i$ satisfy

$Du = \Psi\xi\Phi^{-1}$; so in the fixed basis α_i (where $\Phi_k^i = e_k^i$) Prop 3.4.2 gives

$$a^{ij}(Du) = \Phi_k^i \Phi_l^j a^{kl}(\xi) = e_k^i e_l^j \tilde{a}^{kl}(\sigma_1, \dots, \sigma_n)$$

where we can write $\tilde{a}^{kl}(\sigma_1, \dots, \sigma_n) := \alpha_i^k \alpha_j^l a^{ij}(\xi)$ because the σ_i uniquely determine the ξ . Thus we have reduced the problem to studying the pointwise implications of the invariance. First let $k \neq l$ and define reflections $\Phi(\alpha_i) = (-1)^{\delta_{ik}} \alpha_i$, $\Psi(\beta_i) = (-1)^{\delta_{ik}} \beta_i$. Then $\Psi\xi = \xi\Phi$ and $\Phi = \Phi^{-1}$, so the invariance conditions give (again in α_i coordinates)

$$a^{kl}(\xi) = a^{kl}(\Psi\xi) = a^{kl}(\xi\Phi) = \Phi_i^k \Phi_j^l a^{ij}(\xi) = -a^{kl}(\xi),$$

so $a^{kl}(\xi) = 0$; i.e. \tilde{a} must be diagonal. For three distinct indices k, l, i , if we let Φ be the linear isometry that interchanges α_k and α_l while fixing all other α_i then the right-equivariance implies $a^{ii}(\xi) = a^{ii}(\xi\Phi)$; i.e. \tilde{a}^{ii} is symmetric in all its arguments excluding σ_i . Finally, let Φ swap α_i and α_k , which gives $\tilde{a}^{ii}(\dots) = \tilde{a}^{kk}(\sigma_i \leftrightarrow \sigma_k)$; so $F(\sigma_k; \sigma_1, \dots, \widehat{\sigma_k}, \dots, \sigma_n) = \tilde{a}^{kk}(\sigma_1, \dots, \sigma_n)$ is well defined. \square

Due to the notational inconveniences involved in expressing everything in terms of the scalar *speed function* F , we will often instead simply work with the matrix-valued function \tilde{a} with the understanding that it is diagonal and satisfies the equivariance property identified above; or the vector-valued $F : (0, \infty)^n \rightarrow (0, \infty)^n$ defined by $F_k(\sigma_1, \dots, \sigma_n) = \tilde{a}^{kk}(\sigma_1, \dots, \sigma_n) = F(\sigma_k; \sigma_1, \dots, \widehat{\sigma_k}, \dots, \sigma_n)$ with the understanding that it is permutation-equivariant.

3.4.4 Regularity Issues

We can now define an invariant flow by simply specifying the speed function F ; but in order for such a flow to have any nice properties, the resulting a^{ij} will need to be regular in some sense. It's clear that F is at least as smooth as a^{ij} , but the converse is not so clear, due to the lack of regularity of eigenvalues and eigenvectors we mentioned above. However, it turns out that the equivariant structure is exactly what we need to avoid these singularities creeping in to the actual coefficients a^{ij} . In order to simplify these arguments, we will consider the coefficients instead as a function $A = (a^{ij})$ of the induced metric $u^*g_N = Du^T Du$ and the (vector-valued) speed F as a function of the corresponding eigenvalues $\lambda_k = \sigma_k^2$ - this is no loss of generality since the squaring maps of both the positive reals and the positive-definite symmetric matrices are diffeomorphisms. From this point of view, the equivariance above reduces to equivariance of $A : \text{Sym}_n^+ \rightarrow \text{Sym}_n^+$ under the natural conjugation action of $O(n)$ on Sym_n^+ and likewise that of $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ under permutations. Since $\exp : \text{Sym}_n \rightarrow \text{Sym}_n^+$ is a equivariant diffeomorphism, we can work with logarithms instead, removing the need to be careful about positivity. Thus we want to show the following:

Proposition 3.4.12. *Identify \mathbb{R}^n with the space of diagonal $n \times n$ matrices. If $F : \mathbb{R}^n \rightarrow$*

\mathbb{R}^n is S_n -equivariant and smooth then its unique extension to an $O(n)$ -equivariant map $A : \text{Sym}_n \rightarrow \text{Sym}_n$ is smooth.

We will postpone the proof for just a little while. The idea¹ is to write F in the form

$$F_j(\lambda) = \sum_{k=1}^n f_k(\lambda) \lambda_j^{k-1} \quad (3.4.3)$$

where f_k are smooth symmetric functions. Since power maps $\lambda_j \mapsto \lambda_j^k$ of eigenvalues are induced by power maps $X \mapsto X^k$ of matrices, this will reduce the problem to the regularity of the invariant scalars f_k , which is well-known.

The linear system (3.4.3) can be written using matrices as

$$\begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \cdots & \lambda_1^{n-1} \\ \vdots & & \vdots \\ 1 & \cdots & \lambda_n^{n-1} \end{bmatrix}}_{V(\lambda)} \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

Here $V(\lambda)$ is the well-known Vandermonde matrix, which has determinant $\mathcal{V} = \prod_{i < j} (\lambda_j - \lambda_i)$. Thus when the eigenvalues are all distinct, $V(\lambda)$ is invertible and the equation (3.4.3) has a unique solution f . We can write the solution explicitly using Cramer's rule: we have

$$f_k(\lambda) = \frac{\mathcal{V}_k}{\mathcal{V}}$$

where $\mathcal{V}_k = \det V_k$ is the determinant of the matrix

$$V_k(\lambda) = \begin{bmatrix} 1 & \cdots & \lambda_1^{k-1} & F_1(\lambda) & \lambda_1^{k+1} & \cdots & \lambda_1^{n-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & \cdots & \lambda_n^{k-1} & F_n(\lambda) & \lambda_n^{k+1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

obtained from $V(\lambda)$ by replacing the k^{th} column with $F(\lambda)$. Looking at this matrix, we can see from the equivariance of F that permuting the eigenvalues λ has the same result as permuting the rows of $V_k(\lambda)$; i.e. $V_k : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ is equivariant under the row-permuting action of S_n . From this and basic properties of the determinant we can draw two useful conclusions about \mathcal{V}_k :

- $\mathcal{V}_k(\lambda)$ is alternating under the action of the symmetric group (i.e. $\mathcal{V}_k(\pi\lambda) = \text{sgn}(\pi) \mathcal{V}_k(\lambda)$); and thus
- $\mathcal{V}_k(\lambda)$ vanishes whenever λ contains a repeated eigenvalue.

Since F is smooth and the determinant is a polynomial, \mathcal{V}_k and \mathcal{V} are smooth everywhere; so the coefficients f_k are smooth on the unexceptional set where \mathcal{V} is positive. To show they can be extended to global smooth functions, we need a basic smooth division theorem:

¹This line of reasoning was suggested to me by [Kno].

Lemma 3.4.13 (Hadamard's Lemma). *If $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth function vanishing on the kernel of a linear function $\ell : \mathbb{R}^n \rightarrow \mathbb{R}$, then there is a smooth function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $g = \varphi\ell$.*

Proof. Choose linear coordinates x_1, \dots, x_n such that $\ell = x_1$, and note that the vanishing condition implies

$$g(x) = \int_0^1 \frac{d}{dt} (g(tx_1, x_2, \dots, x_n)) dt.$$

Expanding this with the chain rule we get

$$g(x) = \int_0^1 x_1 (\partial_1 g)(tx_1, x_2, \dots, x_n) dt,$$

and thus

$$\varphi(x) = \int_0^1 (\partial_1 g)(tx_1, x_2, \dots, x_n) dt$$

satisfies $g = \varphi x_1 = \varphi\ell$. Since g is smooth and thus all its derivatives are locally bounded, φ is also smooth. \square

We are now ready to extend the solutions of (3.4.3):

Lemma 3.4.14. *There are smooth symmetric functions $f_k : \mathbb{R}^n \rightarrow \mathbb{R}$ such that (3.4.3) is satisfied.*

Proof. We know \mathcal{V} can be written as the product $\prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i)$, so enumerating these $N = \binom{n}{2}$ linear factors $\lambda_j - \lambda_i$ as $\{\ell_m : 1 \leq m \leq N\}$, we can write

$$\mathcal{V} = \prod_{m=1}^N \ell_m.$$

We will show there is a smooth function f_k such that $\mathcal{V}f_k = \mathcal{V}_k$ by using Lemma 3.4.13 once for each of these N linear factors. First, recall that \mathcal{V}_k vanishes on repeated eigenvalues, and thus wherever ℓ_1 does. Applying the lemma, we obtain a smooth $\varphi_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\ell_1\varphi_1 = \mathcal{V}_k$. To smoothly divide φ_1 by ℓ_2 , note that whenever ℓ_2 vanishes we have $\ell_1\varphi_1 = 0$, and thus φ_1 vanishes on $\{\ell_2 = 0, \ell_1 \neq 0\}$. Since this set is dense in $\{\ell_2 = 0\}$ and φ_1 is continuous, we in fact have $\varphi_1 = 0$ on all of $\{\ell_2 = 0\}$, and thus there is a smooth φ_2 such that $\ell_2\varphi_2 = \varphi_1$. Proceeding in this fashion (each time noting that $\{\ell_m = 0, \ell_{m-1} \neq 0, \dots, \ell_1 \neq 0\}$ is dense in $\{\ell_m = 0\}$) we obtain a factorization

$$\mathcal{V}\varphi_N = \ell_1\ell_2 \dots \ell_N\varphi_N = \mathcal{V}_k,$$

so defining $f_k = \varphi_N$ we have succeeded in finding a smooth extension of $\mathcal{V}_k/\mathcal{V}$. Since both sides of (3.4.3) are continuous functions and the equation is satisfied on the (dense) unexceptional set, the equation is thus satisfied everywhere. To see that $f_k(\lambda)$ is symmetric, note that both \mathcal{V} and \mathcal{V}_k are alternating. \square

We now have all the ingredients we need:

Proof of Proposition 3.4.12. For any given F , let $f_k : \mathbb{R}^n \rightarrow \mathbb{R}$ be the smooth symmetric functions defined above. By Glaeser's "smooth Newton's theorem" [Gla63], there are some smooth $\tilde{f}_k : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f_k = \tilde{f}_k \circ (s_1, \dots, s_n)$, where s_j are the elementary symmetric polynomials. Define the map $A : \text{Sym}_n \rightarrow \text{Sym}_n$ by

$$A(X) = \sum_{k=1}^m \tilde{f}_k(s_1(\lambda(X)), \dots, s_n(\lambda(X))) X^{k-1}.$$

Here the terms $s_j(\lambda(X))$ are the symmetric "matrix invariants" of X , which are smooth functions of X (since they appear as the coefficients of the characteristic polynomial). This is a linear combination of smooth maps and thus smooth, and (as discussed earlier) restricts to F when acting on diagonal matrices. Noting that $\tilde{f}_k \circ s_j(\lambda(X))$ is $O(n)$ -invariant (since the eigenvalues are) and $X \mapsto X^{k-1}$ is $O(n)$ -equivariant, we see that A is equivariant, completing the proof. \square

Corollary 3.4.15. *If $F : (0, \infty)^n \rightarrow (0, \infty)^n$ is smooth and S_n -equivariant, then the coefficients $a : GL(n, \mathbb{R}) \rightarrow \text{Sym}_{n \times n}^+$ defined by*

$$a^{ij}(p) = \sum_k F_k(\sigma_1(p), \dots, \sigma_n(p)) e_k^i e_k^j$$

are smooth. Here e_i, σ_i are the right-singular vectors and singular values of p respectively.

Proof. The singular values of p are the square roots of the eigenvalues of $X = p^T p$ and the right-singular vectors are the corresponding eigenvectors, so we can write a as the composition

$$GL(n, \mathbb{R}) \xrightarrow{p \mapsto p^T p} \text{Sym}_{n \times n}^+ \xrightarrow{\log} \text{Sym}_{n \times n} \xrightarrow{A} \text{Sym}_{n \times n} \xrightarrow{\exp} \text{Sym}_{n \times n}^+$$

where A is the equivariant map on symmetric matrices induced by the smooth equivariant map $(\log \lambda_1, \dots, \log \lambda_n) \mapsto (\log F_1, \dots, \log F_n)(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$ on \mathbb{R}^n . Proposition 3.4.12 tells us that this A is smooth; so since $\sqrt{\cdot}$, \exp , \log are smooth on the domains we are using them on, the composition a^{ij} is too. \square

3.4.5 Derivative Formulae for the Coefficients

For our calculations in the following chapters we will need to know the relationship between derivatives of the coefficients a^{ij} and derivatives of the invariant speed function F , so that we can reduce various conditions on a^{ij} to conditions on F . Since we have now established that a^{ij} is a smooth function of Du , we have a convenient program to compute the derivatives of a^{ij} : first calculate them on the unexceptional set where we have the convenience of smooth singular vectors, and then take limits to get the global formulae on

$GL(n, \mathbb{R})$. From there we can easily obtain the covariant derivatives of a^{ij} by transferring the formulae to manifolds using convenient coordinates.

For this section, let $\mathcal{U} \subset GL(n, \mathbb{R})$ denote the unexceptional set of matrices without repeated singular values, $e_a, v_a \in C^\infty(\mathcal{U}, \mathbb{R}^n)$ the singular vectors and $\sigma_a \in C^\infty(\mathcal{U}, \mathbb{R})$ the singular values. Then we know the diffusion coefficients $a \in C^\infty(\mathcal{U}, \text{Sym}_{n \times n}^+)$ can be written as $a^{ij}(p) = \sum_a F_a(\sigma(p)) e_a^i e_a^j$. On \mathcal{U} this is a combination of smooth functions, so we can differentiate it with the chain and product rules. Since we will want to express things in terms of the geometrically defined frames e, v , it will be convenient to use the Cartan formalism. Thus we will view these as global orthonormal frames of the trivial bundle $\mathcal{U} \times \mathbb{R}^n$ and let $\omega, \tau \in \Gamma(T^*\mathcal{U} \otimes \mathfrak{gl}(n))$ denote their respective connection forms (for now with respect to the canonical flat connection ∂ on the product), which are defined by the equations

$$\partial e_a = \sum_b \omega_a^b \otimes e_b, \quad \partial v_a = \sum_b \tau_a^b \otimes v_b.$$

Since the frames are orthonormal, we in fact know that the connection forms are $\mathfrak{o}(n)$ -valued; so in particular they are zero on the diagonal.

Lemma 3.4.16. *The (off-diagonal components of the) connection forms can be expressed as*

$$\omega_{a\beta}^{bi} = \frac{\sigma_b v_b^\beta e_a^i + \sigma_a v_a^\beta e_b^i}{\sigma_a^2 - \sigma_b^2}, \quad \tau_{a\beta}^{bi} = \frac{\sigma_a v_b^\beta e_a^i + \sigma_b v_a^\beta e_b^i}{\sigma_a^2 - \sigma_b^2}; \quad (3.4.4)$$

the first derivatives of the singular values as

$$\frac{\partial \sigma_a}{\partial p_i^\beta} = e_a^i v_a^\beta \quad (3.4.5)$$

and the second derivatives as

$$\frac{\partial^2 \sigma_a}{\partial p_i^\beta \partial p_j^\gamma} = \sum_b \omega_{a\gamma}^{bj} e_b^i v_a^\beta + \sum_c \sigma_c \omega_{a\beta}^{ci} \omega_{c\gamma}^{aj} - \sum_c \sigma_a \tau_{a\beta}^{ci} \tau_{c\gamma}^{aj}. \quad (3.4.6)$$

Proof. Start by differentiating the defining equation $p_k^\alpha e_a^k = \sigma_a v_a^\alpha$ of the SVD with respect to the matrix component p_i^β , which yields

$$\delta_\beta^\alpha e_a^i + p_k^\alpha \frac{\partial e_a^k}{\partial p_i^\beta} = \frac{\partial \sigma_a}{\partial p_i^\beta} v_a^\alpha + \sigma_a \frac{\partial v_a^\alpha}{\partial p_i^\beta}.$$

We can write this using the connection forms:

$$\delta_\beta^\alpha e_a^i + \sum_c \omega_{a\beta}^{ci} \sigma_c v_c^\alpha = \frac{\partial \sigma_a}{\partial p_i^\beta} v_a^\alpha + \sum_c \sigma_a \tau_{a\beta}^{ci} v_c^\alpha. \quad (3.4.7)$$

Multiplying by some v_b^α , $b \neq a$ and summing over α we get

$$v_b^\beta e_a^i + \sigma_b \omega_{a\beta}^{bi} = \sigma_a \tau_{a\beta}^{bi}.$$

This along with the corresponding equation with a, b switched gives us a 2×2 linear system for the connection forms:

$$\begin{bmatrix} -\sigma_b & \sigma_a \\ \sigma_a & -\sigma_b \end{bmatrix} \begin{bmatrix} \omega_{a\beta}^{bi} \\ \tau_{a\beta}^{bi} \end{bmatrix} = \begin{bmatrix} v_b^\beta e_a^i \\ v_a^\beta e_b^i \end{bmatrix},$$

which (since $\sigma_a \neq \sigma_b$ on \mathcal{U}) we can solve for (3.4.4). Contracting (3.4.7) again, but this time with v_a^α , yields (3.4.5). Differentiating (3.4.7) again and contracting with v_a , the antisymmetry of the connection forms allows us to eliminate all terms involving derivatives of ω, τ , which yields (3.4.6). \square

Now that we have these formulae, we can easily find expressions for the derivatives of \tilde{a}^{ij} on \mathcal{U} : differentiating $a^{ij}(p) = \sum_a F_a(\sigma(p)) e_a^i e_a^j$ with the product rule yields

$$\frac{\partial a^{ij}}{\partial p_k^\alpha} = \sum_{a,b} \left(\frac{\partial F_a}{\partial \sigma_b} \frac{\partial \sigma_b}{\partial p_k^\alpha} e_a^i e_a^j + 2F_a \omega_{a\alpha}^{bk} e_b^{(i} e_a^{j)} \right).$$

Substituting in (3.4.4) and (3.4.5) we find:

Corollary 3.4.17. *On \mathcal{U} , the derivatives of the diffusion coefficients can be expressed as*

$$\frac{\partial a^{ij}}{\partial p_k^\alpha} = \sum_{a,b} \frac{\partial F_a}{\partial \sigma_b} e_b^k v_b^\alpha e_a^i e_a^j + 2 \sum_{a < b} \frac{F_a - F_b}{\sigma_a - \sigma_b} \frac{\sigma_b v_b^\alpha e_a^k + \sigma_a v_a^\alpha e_b^k}{\sigma_a + \sigma_b} e_a^{(i} e_b^{j)}. \quad (3.4.8)$$

In particular, evaluating at the diagonal matrix with distinct entries σ_a and contracting with arbitrary T, S we find

$$\frac{\partial a^{ij}}{\partial p_k^\alpha} T_{ij} S_k^\alpha = \sum_{a,b} \frac{\partial F_a}{\partial \sigma_b} T_{aa} S_b^b + 2 \sum_{a < b} \frac{F_a - F_b}{\sigma_a - \sigma_b} \frac{\sigma_b S_a^b + \sigma_a S_b^a}{\sigma_a + \sigma_b} T_{(ab)}.$$

It's interesting to compare this to Theorem 5.1 of [And], where a very similar formula is found for the *second* derivative of a *scalar* invariant. (Indeed, for symmetric perturbations T, S they agree exactly under the correspondence $a^{ij} = \dot{F}^{ij}$.)

Since we know that the coefficients a^{ij} are smooth on $GL(n, \mathbb{R})$ and that \mathcal{U} is dense in this domain, we can find a general formula for ∂a^{ij} by taking the limit of (3.4.8).

Proposition 3.4.18. *The derivatives of the diffusion coefficients can be globally expressed as*

$$\begin{aligned} \frac{\partial a^{ij}}{\partial p_k^\alpha} = & \sum_{a,b} \frac{\partial F_a}{\partial \sigma_b} e_b^k v_b^\alpha e_a^i e_a^j + 2 \sum_{\substack{a < b \\ \sigma_a \neq \sigma_b}} \frac{F_a - F_b}{\sigma_a - \sigma_b} \frac{\sigma_b v_b^\alpha e_a^k + \sigma_a v_a^\alpha e_b^k}{\sigma_a + \sigma_b} e_a^{(i} e_b^{j)} \\ & + 2 \sum_{\substack{a < b \\ \sigma_a = \sigma_b}} \left(\frac{\partial F_b}{\partial \sigma_b} - \frac{\partial F_b}{\partial \sigma_a} \right) v_a^{(\alpha} e_b^{k)} e_a^{(i} e_b^{j)}. \end{aligned} \quad (3.4.9)$$

Proof. To prove this formula at a given point $p_0 \in GL(n, \mathbb{R})$, we will take the limit along a continuous deformation $p(s)$ of p_0 . Note that we can choose such a $p : [0, \epsilon) \rightarrow GL(n, \mathbb{R})$ satisfying the following conditions:

1. for $s > 0$ we have $p \in \mathcal{U}$; and
2. the $p(s)$ are all diagonalized in the same pair of singular frames.

The first condition means that we can compute the derivative at p_0 by taking the limit of (3.4.9) along p , while the second means we don't have to worry about continuity of e, v . Since the singular values are continuous in s and the frames constant, almost every term of (3.4.8) simply passes through the limit, yielding

$$\frac{\partial a^{ij}}{\partial p_k^\alpha} \Big|_{p_0} = \sum_{a,b} \frac{\partial F_a}{\partial \sigma_b} \Big|_{\sigma(p_0)} e_b^k v_b^\alpha e_a^i e_a^j + 2 \sum_{a < b} \left(\lim_{s \rightarrow 0} \frac{F_a(\sigma(s)) - F_b(\sigma(s))}{\sigma_a(s) - \sigma_b(s)} \right) \frac{\sigma_b v_b^\alpha e_a^k + \sigma_a v_a^\alpha e_b^k}{\sigma_a + \sigma_b} e_a^i e_b^j.$$

When $\sigma_b \neq \sigma_a$, the fraction $(F_a - F_b) / (\sigma_a - \sigma_b)$ also passes through the limit, yielding one of the terms in the second sum of (3.4.9). When $\sigma_b = \sigma_a$, we can use equivariance to write the limit as (reordering the arguments and omitting explicit dependence on s to simplify notation)

$$\lim_{s \rightarrow 0} \frac{F_b(\sigma_b, \sigma_a, \sigma_3, \dots) - F_b(\sigma_a, \sigma_b, \sigma_3, \dots)}{\sigma_a - \sigma_b}.$$

Switching coordinates to $x = \sigma_a - \sigma_b$, $y = \sigma_a + \sigma_b$ the numerator becomes $F_b(x, y, \sigma_3, \dots) - F_b(-x, y, \sigma_3, \dots)$, which we can Taylor expand around the point $(0, y_0 = \sigma_a + \sigma_b, \sigma_3, \dots)$ corresponding to $\sigma(p_0)$. The terms of even order in x all cancel, so we can immediately divide through by the denominator x and substitute $s = 0$, yielding the limit $2\partial_x F_b$. Since $\partial_x = \frac{1}{2}(\partial_{\sigma_b} - \partial_{\sigma_a})$, this is the combination of derivatives in the third sum; and the remaining fraction simplifies using the fact $\sigma_a = \sigma_b$. \square

Now that we have a complete formula for the derivatives of \tilde{a}^{ij} on $GL(n, \mathbb{R})$, it's time to transfer this to the geometric setting. Given a map $u : M \rightarrow N$, fix a point $x_0 \in M$ and choose geodesic normal coordinates x^i about x_0 and y^α about $u(x_0)$ generated by the singular bases of $Du(x_0)$; i.e. such that $\partial_i|_0 = e_i(x_0)$ and $\text{car} \partial_\alpha|_0 = v_\alpha(x_0)$. We will refer to the pair (x, y) of coordinate systems as the *SVD normal coordinates* about x_0 - they will be invaluable to us, not just in this section but in all that follows.

Additionally, define smooth orthonormal frames $E \in \Gamma_{\text{loc}}(FOTM)$, $V \in \Gamma_{\text{loc}}(Fou^*TN)$ by parallel transport of the singular bases along radial geodesics from x_0 . (We use these frames rather than the exact singular frames to avoid the regularity issues with the latter - recall that orthogonal equivariance implies we can choose any orthonormal frames we like to transfer a to the manifold setting.) Then in a neighbourhood of x_0 we have

$$a^{ij}(x) = \sum_{a,b} \tilde{a}^{ab} (V^{-1}(x) Du(x) E(x)) E_a^i(x) E_b^j(x),$$

which we can differentiate to get

$$\nabla_l a^{ij}(x_0) = \sum_{a,b} \frac{\partial \tilde{a}^{ij}}{\partial p_k^\alpha} (\text{diag } \sigma(x_0)) \frac{\partial}{\partial x^l} (V^{-1} Du E)_k^\alpha \Big|_{x_0},$$

since E is parallel at x_0 . Since the Christoffel symbols vanish at the origin and V, E are parallel there, the derivative of $V^{-1} Du E$ there simplifies to $\partial_l (V^{-1} Du E)_k^\alpha = \partial_l \partial_k u^\alpha$, so we have

$$\nabla_l a^{ij}(x_0) = \frac{\partial \tilde{a}^{ij}}{\partial p_k^\alpha} (\text{diag } \sigma(x_0)) \partial_l \partial_k u^\alpha, \quad (3.4.10)$$

which can be written in invariant form as

$$\nabla_l a = \sum_{k,\alpha,a,b} \frac{\partial a^{ab}}{\partial p_k^\alpha} (\text{diag } \sigma) \langle \nabla^2 u(\partial_l, e_k), v_\alpha \rangle e_a \otimes e_b.$$

Substituting (3.4.9) in to (3.4.10) and switching to postfix notation for derivatives of u , we find:

Proposition 3.4.19. *At the origin of an SVD normal coordinate system, we have*

$$\nabla_l a^{ij} = \sum_b \delta_{ij} \frac{\partial F_i}{\partial \sigma_b} u_{bl}^b + \begin{cases} \frac{F_i - F_j}{\sigma_i - \sigma_j} \frac{\sigma_i u_{jl}^i + \sigma_j u_{il}^j}{\sigma_i + \sigma_j} & \text{if } \sigma_i \neq \sigma_j \\ \left(\frac{\partial F_j}{\partial \sigma_j} - \frac{\partial F_j}{\partial \sigma_i} \right) \frac{u_{jl}^i + u_{il}^j}{2} & \text{if } \sigma_i = \sigma_j \end{cases} \quad (3.4.11)$$

where it is understood that F and its derivatives are always evaluated at the singular values of $Du(x_0)$.

As above, we could convert this to an invariant formula by replacing u_{jk}^i with $\langle \nabla^2 u(e_j, e_k), v_i \rangle$; but in practice we will typically be working in the SVD normal coordinates anyway.

3.5 Dual flows

Suppose we have a flow solution $u : [0, T) \rightarrow \text{Diff}(M)$. Since the inverse of a diffeomorphism is a diffeomorphism, a natural question to ask is how the inverse behaves - does the time-wise inverse $u^{-1} : [0, T) \rightarrow \text{Diff}(M)$ also satisfy a PDE? Copious application of the chain rule will allow us to answer this question:

Proposition 3.5.1. *If $u : [0, T) \rightarrow \text{Diff}(M)$ solves an invariant flow with speed function $F(\sigma_1, \dots, \sigma_n)$ then the inverse $v(t) = u^{-1}(t)$ solves the invariant flow with speed function $F(\sigma_1^{-1}, \dots, \sigma_n^{-1}) / \sigma_1^2$.*

Proof. It suffices to prove this at an arbitrary point, so fix SVD normal coordinates about $(x_0, t_0) \in M \times [0, T)$, so that we can use partial derivatives rather than covariant ones.

Starting with $u(v(x, t), t) = \text{id}$ and differentiating, we find the formulae

$$\frac{\partial u^\alpha}{\partial v^i} \frac{\partial v^i}{\partial u^\beta} = \delta_\beta^\alpha \quad (3.5.1)$$

$$\frac{\partial u^\alpha}{\partial v^i} \frac{\partial v^i}{\partial t} + \frac{\partial u^\alpha}{\partial t} = 0 \quad (3.5.2)$$

for the first derivatives. Differentiating (3.5.1) again yields

$$\frac{\partial^2 u^\alpha}{\partial v^i \partial v^j} = - \frac{\partial u^\alpha}{\partial v^k} \frac{\partial^2 v^k}{\partial u^\beta \partial u^\gamma} \frac{\partial u^\beta}{\partial v^i} \frac{\partial u^\gamma}{\partial v^j}.$$

Now, take the flow equation and rewrite it using these formulae:

$$- \frac{\partial u^\alpha}{\partial v^k} \frac{\partial v^k}{\partial t} = -a^{ij} (Dv^{-1}) \frac{\partial u^\alpha}{\partial v^k} \frac{\partial^2 v^k}{\partial u^\beta \partial u^\gamma} \frac{\partial u^\beta}{\partial v^i} \frac{\partial u^\gamma}{\partial v^j},$$

which simplifies to

$$\partial_t v = a^{ij} (Dv^{-1}) (Dv^{-1})_i^\beta (Dv^{-1})_j^\gamma \partial_\beta \partial_\gamma v.$$

Since our coordinates are aligned with the SVD frames and taking singular values commutes with inversion, at the origin this becomes

$$\partial_t v = \hat{a}^{ij} (\sigma_1, \dots, \sigma_n) \partial_i \partial_j v$$

where σ_i denote the singular values of Dv and

$$\hat{a}^{ij} (\sigma_1, \dots, \sigma_n) = \frac{a^{ij} (\sigma_1^{-1}, \dots, \sigma_n^{-1})}{\sigma_i \sigma_j}.$$

Since our coordinates are SVD-aligned (inverting a matrix just swaps its singular frames), this is the formula for the invariant coefficients \tilde{a}^{ij} . \square

We call this equation for v the *dual flow* of the original. This gives us a way to generate new flows that preserve diffeomorphisms (or at least, it will once we know such a flow exists at all), and it can also help us attain estimates - for example, upper bounds on the singular values of a flow translates to lower bounds on the singular values of the dual flow.

Chapter 4

Gradient Estimates and Preserving Diffeomorphisms

Or: *How I Learned To Stop Worrying And Love The Maximum Principle*

As is typical in the analysis of nonlinear PDE, we need some strong estimates on the solutions to rule out the possibility of singularities occurring. Here we are concerned with two kinds of singularities: the usual concerns of our solution or its derivative blowing up to infinity (leading to solutions on a bounded time interval $[0, T)$ which cannot be extended further) are present, but we also want to prevent the derivative Du from becoming singular, so that the flow preserves diffeomorphisms. In the next two chapters we will use the evolution equations we established in Chapter 3 to determine what restrictions are required on the coefficients to get these estimates. This chapter focuses on obtaining C^1 estimates, i.e. control on Du . Such bounds of the appropriate form will allow us to show a flow preserves diffeomorphisms and will also serve as stepping stones to the higher regularity estimates we need for long-time existence.

We saw in the previous chapter that the scalar isometry invariants of Du are exactly the singular values σ_i ; so the geometrically natural way to quantify a “diffeomorphism-preserving estimate” is to preserve a lower bound on the singular values. Thus we will calculate the evolution equation satisfied by the singular values and investigate the structure the speed function F requires in order to obtain a maximum principle. In the case of two-dimensional flat surfaces we find a set of differential inequalities for F that imply the preservation of upper and lower bounds, with $F(\sigma_1, \sigma_2) = (\sigma_1 + \sigma_2)^{-2}$ (corresponding to equation (1.0.1)) jumping out immediately as the most obvious choice of solution. In Appendix B, we show this result can be extended to a Cauchy-Dirichlet problem for the flat disc $\bar{D}^2 \subset \mathbb{R}^2$.

One less than satisfactory outcome of this chapter is the failure to handle curvature, particularly because the higher regularity theory for equation (1.0.1) that we will see in Chapter 5 works on general surfaces. It would be very interesting to understand the

behaviour of the singular values in more generality - for example, if we could prove the analog of Theorem 4.2.3 for the case of constant positive curvature, then we could produce the neat proof of Smale's theorem that we mentioned in the introduction¹.

Throughout this chapter we assume u is a solution of the invariant flow $\partial_t u = a^{ij} \nabla_i \nabla_j u$ where $a^{ij}(Du)$ is of the form (3.4.2). The calculations will become quite dense with quadratic terms in higher derivatives of u , so we will sometimes switch to the compact postfix notation (e.g. $u_i^\alpha = \nabla_i u^\alpha$, $u_{ij}^\alpha = \nabla_j \nabla_i u^\alpha$).

4.1 Evolution of the Singular Values

In §3.4 we saw that the only scalar isometry invariants of the first derivative Du are functions of the singular values σ_i ; so for an invariant flow, these are the natural functions to study in order to control the derivative. Since we want to preserve the condition $\det Du = \prod_i \sigma_i > 0$, it is tempting to look for a maximum principle for the determinant alone. However, this has two drawbacks: firstly, it turns out that finding flows that have such maximum principles is somewhat difficult. More pertinently, even if we found a flow with a two-sided maximum principle for the determinant, this does not provide C^1 estimates on the solution: it is (in principle) possible for the derivative to blow up to infinity in a certain direction while maintaining constant determinant, so we would need to supplement this result with e.g. a maximum principle for ratios of singular values in order to attain C^1 estimates. Thus it behooves us to instead look for control on the singular values directly. First, though, let's get a handle on how the derivative evolves as a whole:

In our analysis of our flows, we will need to know how (functions of) the derivatives of u evolve in time in order to get gradient estimates on u . Thus we should differentiate our flow equation in order to obtain evolution equations for these quantities.

Proposition 4.1.1. *The derivative Du satisfies the evolution equation*

$$(\nabla_t - a^{ij} \nabla_i \nabla_j) u_k^\alpha = \nabla_k a^{ij} \nabla_i \nabla_j u^\alpha + a^{ij} R_{kj i \beta}^{\alpha l} u_l^\beta \quad (4.1.1)$$

where R is the curvature of the natural connection on $\mathfrak{S}^* \otimes u^* TN$ and $\nabla_k a^{ij}$ is the covariant derivative of $X \mapsto a^{ij}(Du(X)) \in \Gamma(\text{Sym}^2 \mathfrak{S})$.

Proof. Recalling Prop 3.1.2, differentiating the flow equation gives

$$\nabla_t Du = \nabla \partial_t u = \nabla \langle a, \nabla^2 u \rangle = (\nabla_k a^{ij} \nabla_i \nabla_j u^\alpha + a^{ij} \nabla_k \nabla_i \nabla_j u^\alpha) dx^k \otimes \partial_\alpha.$$

¹Technically speaking we would still need to determine the long-time behaviour, since our discussion of this matter in Chapter 6 will be quite limited in scope. We also would not quite have a one-step proof: equation (1.0.1) is conformally invariant, so at best it could produce a retraction onto the six-dimensional Möbius group, not the three-dimensional rotation group. While six is a big number to a geometer, it's a lot smaller than the dimension of $\text{Diff}(S^2)$, and there is an elementary argument producing a retraction of the Möbius group onto the orthogonal group; so this would still be a very nice result.

Commuting derivatives gives

$$\nabla_k \nabla_i \nabla_j u^\alpha = \nabla_j \nabla_i \nabla_k u^\alpha + ([\nabla_k, \nabla_j] Du)_i^\alpha = \nabla_j \nabla_i u_k^\alpha + R_{kj i \beta}^{\alpha l} u_l^\beta,$$

so we have the desired formula. \square

We are now ready to study the singular values:

Proposition 4.1.2. *At a point where the singular values σ_a of Du are all distinct, they satisfy*

$$\begin{aligned} P\sigma_a = & \sum_b \frac{\partial a^{kl}}{\partial \sigma_b} u_{ba}^b u_{kl}^a + \sum_{i \neq j} \frac{F_i - F_j}{\sigma_i - \sigma_j} \frac{\sigma_i u_{ja}^i + \sigma_j u_{ia}^j}{\sigma_i + \sigma_j} u_{ij}^a \\ & + a^{ij} \left(-\sigma_a R_{aj}^{Ma}{}_i + \sigma_a \sigma_i \sigma_j R_{aj}^{Na}{}_i + \sum_{b \neq a} \frac{1}{\sigma_b^2 - \sigma_a^2} \left(\sigma_a u_{bi}^a u_{bj}^a + \sigma_a u_{ai}^b u_{aj}^b + 2\sigma_b u_{ai}^b u_{bj}^a \right) \right) \end{aligned} \quad (4.1.2)$$

in the SVD normal coordinates.

Proof. We could prove this by carefully transferring our formulae from (3.4.5) using local coordinates and the chain rule; but we will instead illustrate a different approach: we will use the connection forms of the singular frames on M itself, rather than on $GL(n, \mathbb{R})$, which will make the emergence of the curvature terms quite clear. Start by differentiating the defining equation $Du(e_a) = \sigma_a v_a$:

$$(\nabla_i Du)(e_a) + Du(\nabla_i e_a) = \partial_i \sigma_a v_a + \sigma_a \nabla_i v_a.$$

Letting $\omega, \tau \in \Gamma(T^*M \otimes \mathfrak{so}(n))$ denote the connection forms of the orthonormal frames e, v respectively, we can write this as

$$(\nabla_i Du)(e_a) + \omega_{ai}^b \sigma_b v_b = \partial_i \sigma_a v_a + \sigma_a \tau_{ai}^b v_b. \quad (4.1.3)$$

Taking the inner product with $v_b \neq v_a$ yields the equation

$$\langle (\nabla_i Du)(e_a), v_b \rangle = \sigma_a \tau_{ai}^b - \sigma_b \omega_{ai}^b.$$

Swapping a, b and applying the antisymmetry of the connection forms turns this into

$$\langle (\nabla_i Du)(e_b), v_a \rangle = \sigma_a \omega_{ai}^b - \sigma_b \tau_{ai}^b$$

and solving this 2×2 linear system yields the formulae

$$\omega_{ai}^b = (\sigma_a^2 - \sigma_b^2)^{-1} (\sigma_b u_{ai}^b + \sigma_a u_{bi}^a) \quad (4.1.4)$$

$$\tau_{ai}^b = (\sigma_a^2 - \sigma_b^2)^{-1} (\sigma_a u_{ai}^b + \sigma_b u_{bi}^a) \quad (4.1.5)$$

where the derivatives of u are covariant and we have switched to postfix notation in the SVD normal coordinates. (Of course these formulae really hold only for $a \neq b$; but we know the diagonal components are zero because the frames are orthonormal.) Instead taking the inner product of (4.1.3) with v_a we get $\partial_i \sigma_a = u_{ai}^a$. Differentiating (4.1.3) we get (using the summation convention even when b appears three times)

$$\begin{aligned} (\nabla_i \nabla_j Du)(e_a) + \omega_{ai}^b (\nabla_j Du)(e_b) + \nabla_i \omega_{aj}^b \sigma_b v_b + \omega_{aj}^b \partial_i \sigma_b v_b + \omega_{aj}^b \tau_{bi}^c \sigma_b v_c \\ = \nabla_i \nabla_j \sigma_a v_a + \partial_j \sigma_a \tau_{ai}^b v_b + \partial_i \sigma_a \tau_{aj}^b v_b + \sigma_a \nabla_i \tau_{aj}^b v_b + \sigma_a \tau_{aj}^b \tau_{bi}^c v_c. \end{aligned}$$

Taking the inner product with v_a we can extract

$$\nabla_i \nabla_j \sigma_a = u_{aji}^a + \omega_{ai}^b u_{bj}^a + \omega_{aj}^b \tau_{bi}^a \sigma_b - \tau_{aj}^b \tau_{bi}^a \sigma_a;$$

so

$$P\sigma_a = \partial_t \sigma_a - a^{ij} \nabla_i \nabla_j \sigma_a = (\nabla_t - a^{ij} \nabla_i \nabla_j) \nabla_a u^a - a^{ij} (\omega_{ai}^b u_{bj}^a + \omega_{aj}^b \tau_{bi}^a \sigma_b - \tau_{aj}^b \tau_{bi}^a \sigma_a).$$

Combining this with (4.1.1) we get

$$P\sigma_a = \nabla_a a^{ij} u_{ij}^a + a^{ij} (R_{aj i \beta}^{\alpha l} u_l^\beta - \omega_{ai}^b u_{bj}^a - \omega_{aj}^b \tau_{bi}^a \sigma_b + \tau_{aj}^b \tau_{bi}^a \sigma_a).$$

Substituting (4.1.4), (4.1.5), (3.4.11), Proposition 3.1.3 and simplifying yields the desired formula. \square

4.1.1 Analysis in Two Dimensions

Assume now that $n = 2$, so that evolution equation derived above becomes

$$\begin{aligned} P\sigma_1 = \frac{\partial a^{kl}}{\partial \sigma_b} u_{b1}^b u_{kl}^1 + 2 \frac{F_2 - F_1}{\sigma_2 - \sigma_1} \frac{\sigma_2 u_{11}^2 + \sigma_1 u_{21}^1}{\sigma_1 + \sigma_2} u_{12}^1 \\ + a^{ij} \left(-\sigma_1 R_{1j}^{M1} + \sigma_1 \sigma_i \sigma_j R_{1j}^{N1} + \frac{1}{\sigma_2^2 - \sigma_1^2} (\sigma_1 u_{2i}^1 u_{2j}^1 + \sigma_1 u_{1i}^2 u_{1j}^2 + 2\sigma_2 u_{1i}^2 u_{2j}^1) \right). \end{aligned}$$

To preserve lower bounds on the singular values, the usual maximum principle approach suggests we look to satisfy the condition $P\sigma_1 \geq 0$ whenever $d\sigma_1 = 0$ and $\sigma_2 > \sigma_1$. (If this reasoning is unfamiliar, revise section 2.3.2 - we can assume $d\sigma_1 = 0$ by modifying the parabolic operator with appropriate coefficients b^i .) Note that we are not handling the singular case $\sigma_1 = \sigma_2$ - it turns out that no extra conditions are required to handle this, but we will put off formally verifying this until §4.2, where we will encapsulate both cases in a single argument.

Substituting $a^{kl} = \delta^{kl} F_k$ and $u_{1i}^1 = 0$, the reaction term becomes

$$P\sigma_1 = \dot{F}_2^2 u_{21}^2 u_{22}^1 + \frac{F_1}{\sigma_2^2 - \sigma_1^2} \sigma_1 u_{11}^2 u_{11}^2 \\ + F_2 \left(-\sigma_1 \kappa_M + \sigma_1 \sigma_2^2 \kappa_N + \frac{1}{\sigma_2^2 - \sigma_1^2} (\sigma_1 u_{22}^1 u_{22}^1 + \sigma_1 u_{12}^2 u_{12}^2 + 2\sigma_2 u_{12}^2 u_{22}^1) \right),$$

where \dot{F}_k^l is the derivative of F_k with respect to its l^{th} argument. As we know that the vector-valued F_k is determined by the scalar speed function F , we will now start writing everything in terms of F - this will remove the hidden equivariance and thus help us determine the class of F we are interested in. Since (at least if we want to use an elementary maximum principle argument to get a time-independent bound) we have no local relationship between the first and second derivatives, we can split this in to two contributions that must be independently non-negative: the “flat” term

$$Q_f = \dot{F}^1(\sigma_2, \sigma_1) u_{12}^2 u_{22}^1 + \frac{F(\sigma_1, \sigma_2)}{\sigma_2^2 - \sigma_1^2} \sigma_1 u_{11}^2 u_{11}^2 \\ + \frac{F(\sigma_2, \sigma_1)}{\sigma_2^2 - \sigma_1^2} (\sigma_1 u_{22}^1 u_{22}^1 + \sigma_1 u_{12}^2 u_{12}^2 + 2\sigma_2 u_{12}^2 u_{22}^1) \quad (4.1.6)$$

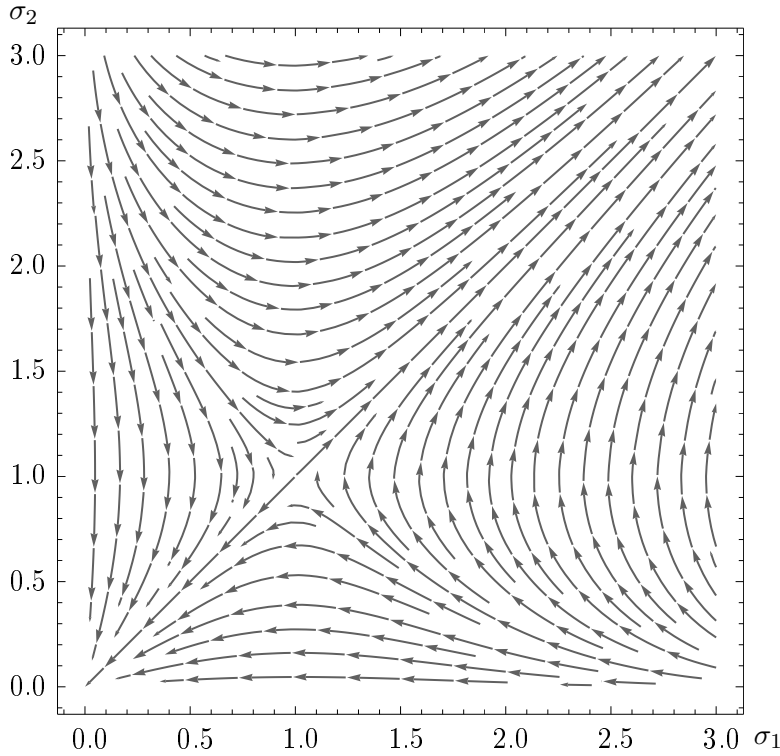
and the curvature term

$$Q_K = \sigma_1 F(\sigma_2, \sigma_1) (\sigma_2^2 \kappa_N - \kappa_M). \quad (4.1.7)$$

This curvature term is very problematic - we can easily choose σ_2 to violate the required inequality $\sigma_2^2 \kappa_N - \kappa_M \geq 0$ unless $\kappa_N \geq 0$ and $\kappa_M \leq 0$; and of course when u is a diffeomorphism of surfaces, the Gauss-Bonnet formula tells us the only way these can both be true for arbitrary σ_i is if both M and N are flat.

It’s nonetheless interesting to attempt to understand this curvature term. When $M = N$ is the round sphere, we get $Q_K = F\sigma_1(\sigma_2^2 - 1)$, which we can interpret as a lowest-order reaction term in the evolution equation for σ_1 . If we neglect the D^2u terms for now, then we can think of the pair (σ_1, σ_2) as satisfying a reaction-diffusion system with reaction vector field $(\dot{\sigma}_1, \dot{\sigma}_2) = F(\sigma_1(\sigma_2^2 - 1), \sigma_2(\sigma_1^2 - 1))$, whose trajectories are illustrated in Figure 4.1.1. We see that this “force due to curvature” pushes the singular values closer together, but also generically pushes their magnitude to zero or infinity. This first property is nice, and for the flow (1.0.1) does still hold when we add the D^2u terms back in - some calculations similar to the ones we are in the middle of show that at a point where $d(\sigma_2/\sigma_1) = 0$, the sign of $P(\sigma_2/\sigma_1)$ is equal to the sign of $\sigma_1 - \sigma_2$; so upper bounds on the eccentricity $\sigma_{\max}/\sigma_{\min}$ are preserved. However, the second property (the tendency of the vector field towards zero and infinity along the diagonal) kills our hopes of preserving upper/lower bounds on the singular values themselves. In the hyperbolic case, the direction of this vector field would be reversed, so things could go wrong in a different fashion: the singular values might fly off to infinity along the hyperbola $\sigma_1\sigma_2 = 1$.

One possible line of attack is to preserve a time-dependent lower bound: while the

Figure 4.1.1: Flow plot for the vector field $\dot{\sigma}$.

stream lines in Figure 4.1.1 rule out preserving a constant bound, it's possible that the speed at which we flow along them slows down fast enough that we never actually reach degeneracy. We can adjust the speed by changing the defining function F : for example if $F = 1$, then along the diagonal near $\sigma_1 = \sigma_2 = 0$ we have $\dot{\sigma} = \sigma^3 - \sigma \approx -\sigma$; so we could hope to preserve an exponentially decaying lower bound. Unfortunately, we will see soon that in order to make Q_f non-negative, F must necessarily vary something like σ^{-2} , which will rule out even a time-dependent bound.

Resigning ourselves to flatness for the sake of attaining a maximum principle, we now investigate the term Q_f . Noting that Q_f is a quadratic form in the components $\{u_{22}^1, u_{11}^2, u_{12}^2\}$, we make the change of the basis (and notation) to

$$\begin{aligned} w &= \frac{1}{2} (u_{12}^2 + u_{22}^1) \\ z &= \frac{1}{2} (u_{12}^2 - u_{22}^1) \\ x &= u_{11}^2, \end{aligned} \tag{4.1.8}$$

diagonalizing the reaction term:

$$Q_f = \left[\frac{F(\sigma_1, \sigma_2) \sigma_1}{\sigma_2^2 - \sigma_1^2} \right] x^2 + \left[\dot{F}^1(\sigma_2, \sigma_1) + \frac{2(\sigma_1 + \sigma_2)}{\sigma_2^2 - \sigma_1^2} F(\sigma_2, \sigma_1) \right] w^2 \\ + \left[-\dot{F}^1(\sigma_2, \sigma_1) - \frac{2(\sigma_2 - \sigma_1)}{\sigma_2^2 - \sigma_1^2} F(\sigma_2, \sigma_1) \right] z^2.$$

Thus we find a necessary condition for the flow to preserve lower bounds on the singular values:

$$x > y \implies -\frac{2}{x-y} \leq \frac{\partial \log F(x, y)}{\partial x} \leq -\frac{2}{x+y} \quad (4.1.9)$$

Likewise, to preserve upper bounds we need $Q_f \leq 0$ whenever $\sigma_1 > \sigma_2$, which reduces to

$$x < y \implies -\frac{2}{y+x} \leq \frac{\partial \log F(x, y)}{\partial x} \leq \frac{2}{y-x}. \quad (4.1.10)$$

Note that the upper bound in (4.1.10) is positive, so this condition is much easier to satisfy - this is expected, since even harmonic map heat flow ($F = 1$) preserves upper bounds.

It's important to remember that we don't really have a maximum principle yet - the condition (4.1.9) is enough to ensure $P\sigma_1 \geq 0$ when $\sigma_1 < \sigma_2$, but we have not yet ruled out the possibility of a new minimum being attained at a point where $\sigma_1 = \sigma_2$. It will turn out, however, that the conditions (4.1.9), (4.1.10) are sufficient when we do things properly in Theorem 4.2.3; so before getting to the technical details we will see some examples of flows satisfying these conditions.

4.1.2 Examples in Two Dimensions

By informally integrating both sides of (4.1.9), we see that we should expect any flow preserving lower bounds to have coefficients that are roughly homogeneous of degree -2 in the singular values. Unfortunately this seems to rule out choosing a speed function F that both satisfies this requirement and makes the flow of the vector field depicted in Figure 4.1.1 exist for all time: if the singular values are near zero, they would approximately obey $\sigma' = -\kappa/\sigma$, which degenerates to zero in finite time. Thus even a time-dependent lower bound on the singular values seems difficult to achieve in the presence of curvature, justifying our pragmatic restriction to $\kappa = 0$.

The most obvious way to satisfy both conditions (4.1.9), (4.1.10) is to satisfy all four inequalities irrespective of the sign of $x - y$. Just looking at these inequalities we see that this is only possible if F solves the differential equation

$$\frac{\partial \log F(x, y)}{\partial x} = -\frac{2}{x+y},$$

which has general solution $F(x, y) = C(x+y)^{-2}$. Thus the flow

$$\frac{\partial u}{\partial t} = \frac{\Delta u}{(\sigma_1 + \sigma_2)^2} = \frac{\Delta u}{|Du|^2 + 2 \det Du} \quad (1.0.1 \text{ revisited})$$

is the first flow we find that preserves both upper and lower bounds on the singular values. (Choosing a different constant of integration just rescales the flow solutions in time, so there's no loss of generality fixing $C = 1$.)

Another solution is $F(x, y) = (x^2 + y^2)^{-1}$, which yields a flow that looks simpler when expressed without the help of the singular values:

$$\frac{\partial u}{\partial t} = \frac{\Delta u}{|Du|^2} \quad (4.1.11)$$

However, we will see in the next chapter that $(\sigma_1 + \sigma_2)^2$ is actually a better denominator to have when it comes to getting higher regularity.

One interesting example that preserves lower bounds but not upper bounds is the speed function $F(x, y) = 1/(x(x + y))$, which generates the anisotropic flow

$$\frac{\partial u}{\partial t} = \frac{1}{(\sigma_1 + \sigma_2)} \left(\frac{1}{\sigma_1} \nabla_{e_1} \nabla_{e_1} u + \frac{1}{\sigma_1} \nabla_{e_2} \nabla_{e_2} u \right).$$

In the case where $N = M$ and the map u is locally area-preserving (i.e. $\sigma_1 \sigma_2 = 1$), this equation is equivalent to mean curvature flow of the graph of u (thought of as a surface in $M \times M$). In the constant-curvature case, it turns out that the area-preserving condition is preserved by the flow. (Observe in Figure 4.1.1 that the vector field $\dot{\sigma}$ is tangent to the hyperbola $\sigma_1 \sigma_2 = 1$, one of the necessary conditions for this preservation.) This flow was studied (using the methods of Lagrangian mean curvature flow) in the paper [Wan01], where a complete result (for this particular class of diffeomorphisms) was obtained.

4.1.3 Difficulties in Higher Dimensions

We will now take a quick detour to highlight the difficulties in generalizing this kind of maximum principle to higher dimensions. Starting with Proposition 4.1.2, let's assume $R = 0$ and $a = 1$, which yields (using the vector-valued speed function $F_k = F(\sigma_k; \dots)$) a generalization of our Q_f in the previous section:

$$\begin{aligned} P\sigma_1 = & \sum_{b,k} \frac{\partial F_k}{\partial \sigma_b} u_{ba}^b u_{kk}^a + 2 \sum_{i < j} \frac{F_i - F_j}{\sigma_i - \sigma_j} \frac{\sigma_i u_{j1}^i + \sigma_j u_{i1}^j}{\sigma_i + \sigma_j} u_{ij}^1 \\ & + \sum_k F_k \sum_{b \neq 1} \frac{1}{\sigma_b^2 - \sigma_1^2} \left(\sigma_1 u_{bk}^1 u_{bk}^1 + \sigma_1 u_{1k}^b u_{1k}^b + 2\sigma_b u_{1k}^b u_{bk}^1 \right). \end{aligned}$$

Completing the square in the last term and assuming $n = 3$ along with the critical point condition $u_{1i}^1 = 0$, we can write this as

$$P\sigma_1 = \sum_{b,k} \frac{\partial F_k}{\partial \sigma_b} u_{b1}^b u_{kk}^1 + 2 \frac{F_2 - F_3}{\sigma_2 - \sigma_3} \frac{\sigma_2 u_{31}^2 + \sigma_3 u_{21}^3}{\sigma_2 + \sigma_3} u_{23}^1 + \sum_k F_k \sum_{b=2,3} \left(\frac{\sigma_1 (u_{bk}^1 + u_{1k}^b)^2}{\sigma_b^2 - \sigma_1^2} + \frac{2u_{1k}^b u_{bk}^1}{\sigma_b + \sigma_1} \right).$$

In order to preserve lower bounds on the singular values with a maximum principles we must choose $F(\sigma)$ such that $\sigma_1 < \sigma_2 \leq \sigma_3$ implies $P\sigma_1 \geq 0$ whenever $u_{1i}^1 = 0$. As in the two-dimensional case, by carefully prescribing $\partial \log F$ we should be able to control the terms with repeated indices (e.g. $u_{12}^2 u_{33}^1$). The key terms to focus on are those with no repeated indices, which have been introduced both in the anisotropy term (arising from the difference $F_2 - F_3$) and the final term. Note in particular that these two different classes of components of Du are never multiplied together, so we can analyse their sign separately. Isolating the “fully mixed” terms we have

$$Q_m = 2 \frac{F_2 - F_3}{\sigma_2^2 - \sigma_3^2} (\sigma_2 u_{31}^2 + \sigma_3 u_{21}^3) u_{23}^1 + \sum_{k \neq 1} F_k \sum_{\substack{b \neq 1 \\ b \neq k}} \left(\frac{\sigma_1 (u_{bk}^1 + u_{1k}^b)^2}{\sigma_b^2 - \sigma_1^2} + \frac{2u_{1k}^b u_{bk}^1}{\sigma_b + \sigma_1} \right).$$

Choosing an equivariant F to make this unconditionally non-negative seems impossible - the lack of ∂F terms available means we have very little control. In some sense the “nice” assumption of isometry invariance is stabbing us in the back here: it implies that the terms arising from the derivatives of F always come with repeated indices in the SVD frames, so they can’t be leveraged to kill off the omnipresent mixed-index terms in the same way we chose F in the two-dimensional case.

4.2 Maximum Principle for the Induced Metric

In §4.1 we laid the groundwork for a maximum principle for the singular values, and in particular found the necessary conditions (4.1.9) and (4.1.10) in two dimensions. In order to show these conditions are in fact sufficient, however, we need to rule out the possibility of both singular values reaching a new minimum together. This introduces a difficulty: the singular values are not necessarily differentiable functions of Du (and thus of space and time) when they are equal [Kat82]. It is possible to handle these points as a special case by carefully analysing something like $\sigma_1 \sigma_2$, but for the purpose of a neater proof we will avoid this approach.

Instead, we take a page from Hamilton’s book (well, really his paper [Ham82]): we look for a tensor maximum principle for the *induced metric* $h = u^*g$, which is a symmetric bilinear form on \mathfrak{S} given in coordinates by $h_{ij} = u_i^\alpha u_j^\beta g_{\alpha\beta}$, or perhaps more familiarly as the Gram matrix $h = Du^T Du$, where the “transpose” is really the adjoint with respect to the Riemannian metric of N . Remembering the construction of the SVD we gave in §3.4, we see that the singular values of Du are the square roots of the eigenvalues of h . Thus preserving bounds $\lambda_- \leq \sigma_i \leq \lambda_+$ on the singular values is equivalent to preserving the inequalities

$$\lambda_-^2 g \leq h \leq \lambda_+^2 g$$

of bilinear forms.² To do so, we use a sharp version of Hamilton's maximum principle that can be found in [And, Theorem 3.2]:

Theorem 4.2.1. *Let $S \in \Gamma(\text{Sym}^2 \mathfrak{S}^*)$ satisfy the evolution equation*

$$\partial_t S = a^{kl} \nabla_k \nabla_l S + b^k \nabla_k S + Q$$

where $a \in \Gamma(\text{Sym}^2 \mathfrak{S})$ is positive definite. If $S \geq 0$ at the initial time and

$$Q(w, w) + \sup_{\Gamma} 2a^{kl} (2\Gamma_k^p \nabla_l S_{ip} w^i - \Gamma_k^p \Gamma_l^q S_{pq}) \geq 0 \quad (4.2.1)$$

whenever $S \geq 0$ and $S_{ij} w^i = 0$, then $S \geq 0$ for all time.

The required positivity condition (4.2.1) seems quite arcane at first sight, but it is a consequence of a simple observation: if S is non-negative definite everywhere but $S_X(w, w) = 0$, then for *any* vector field V extending w , the scalar function $S(V, V)$ must have non-negative definite Hessian at X . A calculation shows that this Hessian turns out to depend only upon the first derivative of V at X , which is what Γ denotes. (Thus if one wants to get pedantic, the supremum is taken over $\Gamma \in \mathfrak{S}_X^* \otimes \mathfrak{S}_X$ where X is the basepoint of w .)

Note that the desired bounds on h can be expressed as non-negativity of $S^- = h - \lambda_-^2 g$ and $S^+ = \lambda_+^2 g - h$, so once we know the evolution equation of h we can investigate the assumptions required to apply the theorem.

Proposition 4.2.2. *If $u : M \times [0, T) \rightarrow N$ solves the evolution equation*

$$\partial_t u^\alpha = a^{ij} (Du) \nabla_i \nabla_j u^\alpha,$$

then h satisfies $\partial_t h_{ij} = a^{kl} \nabla_k \nabla_l h_{ij} + Q_{ij}$ where

$$Q_{ij} = 2g_{\alpha\beta} \left(u_{(j}^\beta \nabla_i) a^{kl} \nabla_k \nabla_l u^\alpha + a^{kl} u_{(j}^\beta R_{i)kl}{}^{\alpha a} u_a^\mu - a^{kl} \nabla_l u_i^\alpha \nabla_k u_j^\beta \right).$$

Proof. Remember that we are using the pullback connection for the u^*TN factors, so $g_{\alpha\beta} \in \Gamma(\text{Sym}^2 u^*TN^*)$ is ∇ -parallel (including in the ∂_t direction). First differentiate h in time:

$$\nabla_t h_{ij} = \nabla_t (g_{\alpha\beta} u_i^\alpha u_j^\beta) = 2g_{\alpha\beta} \nabla_t u_i^\alpha u_j^\beta.$$

(Here we are using u_i^α to denote Du .) Now differentiate twice in space:

$$\nabla_k \nabla_l h_{ij} = \nabla_k (2g_{\alpha\beta} \nabla_l u_i^\alpha u_j^\beta) = 2 \left(g_{\alpha\beta} \nabla_l u_{(i}^\alpha \nabla_k u_{j)}^\beta + \nabla_k \nabla_l u_{(i}^\alpha u_{j)}^\beta \right).$$

Putting these together we get

$$\left(\nabla_t - a^{kl} \nabla_k \nabla_l \right) h_{ij} = 2g_{\alpha\beta} u_{(j}^\beta \left(\nabla_t - a^{kl} \nabla_k \nabla_l \right) u_{i)}^\alpha - 2a^{kl} g_{\alpha\beta} \nabla_l u_{(i}^\alpha \nabla_k u_{j)}^\beta.$$

²Equivalently, we are trying to preserve the bounds $\lambda_-^2 \leq \beta(v) \leq \lambda_+^2$ for the "stretch function" β on the unit tangent bundle $UTM = \{v \in TM : |v| = 1\}$ defined by $\beta(v) = h(v, v)$; so an equivalent approach is to apply the scalar maximum principle to this function.

We now need to apply the evolution equation: differentiating $\nabla_t u^\alpha = a^{kl} (Du) \nabla_k \nabla_l u^\alpha$ yields

$$\nabla_i \nabla_t u^\alpha = \nabla_i a^{kl} \nabla_k \nabla_l u^\alpha + a^{kl} \nabla_i \nabla_k \nabla_l u^\alpha.$$

Using the definition of the curvature and the fact that ∇Du is symmetric, we have $\nabla_i \nabla_k \nabla_l u^\alpha = R_{ikl}^{\alpha j} u_j^\beta + \nabla_k \nabla_l \nabla_i u^\alpha$ and thus

$$\left(\nabla_t - a^{kl} \nabla_k \nabla_l \right) u_i^\alpha = \nabla_i a^{kl} \nabla_k \nabla_l u^\alpha + a^{kl} R_{ikl}^{\alpha j} u_j^\beta; \quad (4.2.2)$$

so

$$\left(\nabla_t - a^{kl} \nabla_k \nabla_l \right) h_{ij} = 2g_{\alpha\beta} \left(u_{(j}^\beta \nabla_{i)} a^{kl} \nabla_k \nabla_l u^\alpha + a^{kl} u_{(j}^\beta R_{i)kl}^{\alpha a} u_a^\gamma - a^{kl} \nabla_l u_i^\alpha \nabla_k u_j^\beta \right).$$

Since $\nabla_t = \partial_t$ on TM , this is the desired formula. \square

Note in particular that the term $Q_{ij}^H = -2g_{\alpha\beta} a^{kl} \nabla_l u_i^\alpha \nabla_k u_j^\beta$ is negative-definite, since $Q_{ij}^H v^i v^j$ can be written as the inner product of the definite matrices $-2a^{kl}$ and $\nabla^2 u(v, \partial_l) \cdot \nabla^2 u(v, \partial_k)$. When $a^{ij} = g^{ij}$ and $R = 0$ (i.e. when dealing with harmonic map heat flow between flat manifolds) we have $Q = Q^H$ and thus we can preserve upper bounds on the singular values but not lower bounds. Therefore if we want our flow to preserve lower bounds too, we need to constrain a^{ij} so that $Q_{ij}^{\nabla a} = 2g_{\alpha\beta} u_{(j}^\beta \nabla_{i)} a^{kl} \nabla_k \nabla_l u^\alpha$ along with the supremum term can overpower this negativity. This reinforces the point we made earlier regarding the relative strengths of (4.1.9) and (4.1.10) - in general we should expect many more flows to have C^1 estimates (preserve $h \leq \lambda_+^2 g$) than to preserve diffeomorphisms (preserve $h \geq \lambda_-^2 g$).

We are now almost ready to prove our gradient estimate. Since $S = S^\pm = \pm(\lambda_\pm g - h)$ and g is parallel, S satisfies the evolution equation $PS = \mp Ph$. Suppose we are at the first point where $S_{ij} w^i = 0$ (and thus the singular value associated to w is equal to λ_\pm). The condition (4.2.1) then becomes (in SVD normal coordinates with $e_1 = w$)

$$\mp 2\lambda_\pm \left(\nabla_1 a^{kl} u_{kl}^1 + a^{kl} R_{1kl}^{1a} u_a^\mu \right) \pm 2a^{kl} u_{k1}^\alpha u_{l1}^\alpha + \sup_\Gamma 2a^{kl} (2\Gamma_k^p \nabla_l S_{1p} - \Gamma_k^p \Gamma_l^q S_{pq}) \geq 0. \quad (4.2.3)$$

First let's consider the supremum term, which we will refer to as Q_Γ . Since we know a^{kl} and S_{pq} are diagonal in our coordinates, the term being maximized becomes

$$\sum_{k,p} 2F_k (2\Gamma_k^p \nabla_k S_{1p} - \Gamma_k^p \Gamma_k^p S_{pp}),$$

which is a sum of univariate quadratic polynomials in the variables Γ_k^p . Thus we can find the supremum by maximizing each of them individually. If there is a p such that $S_{pp} = 0$ but $\nabla_k S_{1p} \neq 0$ then we get a nonconstant linear polynomial, so we can immediately

conclude $Q_\Gamma = \infty$ and thus (4.2.3) is satisfied. Otherwise, the polynomial in Γ_k^p is either zero or the nondegenerate quadratic $F_k \Gamma_k^p (2\nabla_k S_{1p} - S_{pp} \Gamma_k^p)$, which has global maximum at $\Gamma_k^p = \nabla_k S_{1p} / S_{pp}$, where it is equal to $F_k (\nabla_k S_{1p})^2 / S_{pp}$. Thus in this case the supremum is

$$Q_\Gamma = 2 \sum_{\substack{k,p \\ S_{pp} \neq 0}} \frac{F_k (\nabla_k S_{1p})^2}{S_{pp}} = \mp \sum_{\substack{k,p \\ \sigma_p \neq \sigma_1}} \frac{2F_k (\sigma_1 u_{kp}^1 + \sigma_p u_{k1}^p)^2}{(\sigma_p^2 - \sigma_1^2)}. \quad (4.2.4)$$

Note that the conditions $S_{pp} = 0$ and $\nabla_k S_{1p} \neq 0$ can also be expressed as $\sigma_p = \sigma_1$ and $u_{kp}^1 + u_{k1}^p \neq 0$ respectively; so the degenerate case of coinciding singular values that gave us pause in the previous section is actually a boon to us here: for almost any choice of second derivatives we get the desired sign without needing to analyse any other terms. On the other hand, when $\sigma_p = \sigma_1$ and $u_{kp}^1 + u_{k1}^p = 0$ so that the corresponding contribution to the supremum vanishes, this strong constraint on the second derivatives should help us get the positivity we need from the term Q^H .

Let

$$Y_{ij} = \begin{cases} \frac{1}{\sigma_i + \sigma_j} \frac{F_i - F_j}{\sigma_i - \sigma_j} & \text{if } \sigma_i \neq \sigma_j \\ \frac{1}{\sigma_i + \sigma_j} \left(\frac{\partial F_j}{\partial \sigma_j} - \frac{\partial F_j}{\partial \sigma_i} \right) & \text{if } \sigma_i = \sigma_j \end{cases}$$

denote the coefficient of $\sigma_i u_{jl}^i + \sigma_j u_{il}^j$ in (3.4.11). The condition (4.2.3) then becomes

$$\begin{aligned} \widehat{Q} := \mp 2\sigma_1 \sum_{b,a} \left(\frac{\partial F_a}{\partial \sigma_b} u_{b1}^b u_{aa}^1 + Y_{ab} (\sigma_a u_{b1}^a + \sigma_b u_{a1}^b) u_{ab}^1 \right) \mp 2\sigma_1 \sum_a F_a R_{1a}^{1j} u_j^\mu \\ \pm 2 \sum_{a,b} F_a u_{a1}^b u_{a1}^b + \sup_\Gamma a^{kl} (2\Gamma_k^p \nabla_l S_{1p} - \Gamma_k^p \Gamma_l^q S_{pq}) \geq 0. \end{aligned} \quad (4.2.5)$$

where we used the zeroth-order condition $\sigma_1 = \lambda_\pm$. To get our first understanding of this expression, we will begin by studying the two-dimensional flat case. Assuming that $n = 2$ and $R = 0$, as in section 4.1.1, the condition (4.2.3) becomes:

Theorem 4.2.3. *Suppose the speed function F satisfies the conditions (4.1.9), (4.1.10). If an evolving map u between flat surfaces (resp. flat surfaces with boundary) satisfies the flow equation $\partial_t u = a^{ij} \nabla_i \nabla_j u$ defined by F , then the extrema of the singular values of Du are attained at the initial time (resp. on the parabolic boundary).*

Proof. Thanks to the discussion in the previous section, we know that the necessary condition is that $\widehat{Q} \geq 0$ whenever $S \geq 0$ and $w = e_1 \in UTM$ satisfies $S_{ij} w^i = 0$. Since these two conditions imply $\nabla_i S_{11} = 2\sigma_1 u_{1i}^1 = 0$, the dimension and curvature assumptions allow us to reduce (4.2.5) to

$$\widehat{Q} = \mp 2 \frac{\partial F_2}{\partial \sigma_2} \sigma_1 u_{21}^2 u_{22}^1 \pm 2 \sum_a F_a u_{1a}^2 u_{1a}^2 + \sup_\Gamma a^{kl} (2\Gamma_k^p \nabla_l S_{1p} - \Gamma_k^p \Gamma_l^q S_{pq}) \geq 0.$$

As discussed earlier, the supremum is easily handled by a quick case analysis: if we are at

an exceptional point where $\sigma_1 = \sigma_2$ then either

1. at least one of u_{11}^2 and $u_{22}^1 + u_{21}^2$ are non-zero, so $Q_\Gamma = +\infty$ (as described in the previous section) and we are done; or
2. $u_{11}^2 = 0$ and $u_{22}^1 + u_{21}^2 = 0$ (so that Q_Γ vanishes), so that e_2 is also a spatial minimizer of the quadratic form S and thus $\nabla_i S_{22} = 0$. Putting these together we must in fact have $u_{11}^2 = u_{22}^1 = u_{21}^2 = 0$, so $\hat{Q} = 0$ and we are done.

At an unexceptional point ($\sigma_1 \neq \sigma_2$) we know Q_Γ takes the form (4.2.4), so we have

$$\begin{aligned}\hat{Q} &= \mp 2 \frac{\partial F_2}{\partial \sigma_2} \sigma_1 u_{21}^2 u_{22}^1 \pm 2 \sum_a F_a u_{1a}^2 u_{1a}^2 \mp 2 \sum_a \frac{F_2 (\sigma_1 u_{22}^1 + \sigma_2 u_{12}^2)^2 + F_1 (\sigma_2 u_{11}^2)^2}{\sigma_2^2 - \sigma_1^2} \\ &= \mp 2 \frac{\partial F_2}{\partial \sigma_2} \sigma_1 u_{21}^2 u_{22}^1 \mp 2 \sum_a \frac{F_2 ((\sigma_1 u_{22}^1)^2 + 2\sigma_1 \sigma_2 u_{22}^1 u_{12}^2 + (\sigma_1 u_{12}^2)^2) + F_1 (\sigma_1 u_{11}^2)^2}{\sigma_2^2 - \sigma_1^2}.\end{aligned}$$

Comparing this to (4.1.6) we see that this is exactly $\hat{Q} = \mp 2\sigma_1 Q_f$, so this case is handled by the argument in section 4.1.1: when the sign is $+$ we have $\sigma_1 > \sigma_2$ and thus (4.1.10) implies $Q_f \leq 0$, yielding $\hat{Q} \geq 0$. On the other hand, when the sign is $-$ we have $\sigma_1 < \sigma_2$ and thus (4.1.9) yields $Q_f \geq 0$ and in turn $\hat{Q} \geq 0$. \square

This generic reduction to the maximum principle for the singular values is no accident:

Proposition 4.2.4. *At a point where the singular values are distinct, $\sigma_1 = \lambda_\pm$ and $d\sigma_1 = 0$, we have $\hat{Q} = \mp 2\sigma_1 P\sigma_1$.*

Proof. Since Du is on the unexceptional set, we can substitute in the nice expressions of Y_{ab} and Q_Γ to find

$$\begin{aligned}\hat{Q} &:= \mp 2\sigma_1 \sum_{b,a} \left(\frac{\partial F_a}{\partial \sigma_b} u_{b1}^b u_{aa}^1 + \frac{F_a - F_b}{\sigma_a^2 - \sigma_b^2} (\sigma_a u_{b1}^a + \sigma_b u_{a1}^b) u_{ab}^1 \right) \mp 2\sigma_1 \sum_a F_a R_{1a\mu}^{1j} u_j^\mu \\ &\quad \mp 2 \sum_k \sum_{p \neq 1} \left(\frac{F_k (\sigma_1 u_{kp}^1 + \sigma_p u_{k1}^p)^2}{\sigma_p^2 - \sigma_1^2} - 2F_k u_{k1}^p u_{k1}^p \right).\end{aligned}$$

Comparing this with (4.1.2), we see immediately that the terms coming from ∇a agree, and likewise the curvature terms; so we just need to focus on the term on the second line. Moving the $2F_k u_{k1}^p u_{k1}^p$ to the top of the fraction, expanding and simplifying, this term becomes

$$\mp \sum_k F_k \sum_{p \neq 1} \frac{\sigma_1 (u_{bk}^1)^2 + 2\sigma_b u_{1k}^b u_{bk}^1 + \sigma_1 (u_{1k}^b)^2}{\sigma_b^2 - \sigma_1^2},$$

which is as desired $\mp 2\sigma_1$ times the corresponding term in $P\sigma_1$. \square

Thus the tensor maximum principle doesn't improve our chances of finding a nice flow - it just provides a neat way to handle the exceptional points.

As a final remark for this chapter, the case of surfaces with boundary could be a very interesting direction to take these ideas in: If M has boundary, the maximum principle Theorem 4.2.3 shows that the extra work required to preserve bounds on the singular values is simply to preserve these bounds along ∂M . With the right boundary conditions for the flow, this is achievable using barriers: see Appendix A.2 for an example.

Chapter 5

Hölder Gradient Estimates and Higher Regularity

In the previous chapter we derived estimates on the first derivatives of our solution using the maximum principle. While such bounds are enough to show a flow preserves diffeomorphisms so long as the solution exists, and are certainly necessary to achieve long-time existence, they are far from sufficient for the latter. In order to rule out the formation of a singularity past which the classical flow cannot continue, we need uniform bounds on *all* derivatives, so that a solution on $[0, T)$ can always be smoothly continued to time T . (We will see the details of this argument in §6.2).

In many heat-type geometric flows, uniform control of the first derivative or two is enough to get bounds on all derivatives using a maximum principle technique in the tradition of early work by Bernstein [Ber12]. For example, in harmonic map heat flow between manifolds of bounded geometry (C^1 bounds on the curvature of both domain and target), bounds on $|Du|$ are enough; while for Ricci flow bounds on the curvature tensor suffice. (See [AH11], Proposition 2.6 and Theorem 7.1 respectively.)

Unfortunately, the nonlinearity we have introduced in our coefficients $a^{ij}(Du)$ rules out such an argument: for example the evolution equation of $|Du|^2 + t|D^2u|^2$ would pick up terms involving second derivatives of a^{ij} and thus third derivatives of u ; so the Bernstein method cannot help us. Instead, we can exploit the Schauder theory along with the quasilinear structure of our flow. Recall that the Schauder estimate (Proposition 2.4.19) tells us that parabolic PDE with C^α coefficients have $C^{2,\alpha}$ solutions. If u is $C^{2,\alpha}$ and a^{ij} is nice, then we will in fact have $a^{ij}(Du) \in C^{1,\alpha}$: the coefficients gain an extra derivative of regularity for free. But wait, there's more: this extra regularity then implies that the coefficients of the evolution equation of Du are C^α , so the same trick yields $Du \in C^{2,\alpha}$ and thus $a^{ij}(Du) \in C^{2,\alpha}$. Iterating this argument we see that the solution “pulls itself up by its bootstraps” all the way to C^∞ - all we need to do is provide the first little boost to get it to C^α .

Thus, our goal is now a Hölder estimate for the flow coefficients (or alternatively the full gradient Du). In the case of a scalar parabolic PDE, modest structure conditions are

enough to get Hölder gradient estimates - see e.g. [Lie96]. Since we are working with a *system* of PDEs, we have no such general theory to lean on; so we need to exploit the particular form of our system to have any chance of obtaining Hölder estimates for the coefficients.

We will start by investigating a special class of flows and initial data on \mathbb{R}^n for which we can reduce the evolution to a scalar PDE, and then use the insights we gain from this to choose a flow on \mathbb{R}^2 for which we can construct a local Hölder estimate for general data. We will then show how this estimate can be generalized to work for arbitrary surfaces.

5.1 Exact Diffeomorphisms

In this section we will restrict ourselves to considering diffeomorphisms of \mathbb{R}^n with symmetric first derivative matrices $u_i^\alpha = u_\alpha^i$; i.e. that are gradients $u^\alpha = \partial_\alpha \phi$ when interpreted as vector fields. We will call such a diffeomorphism *exact*, and the corresponding scalar function ϕ (which is uniquely determined up to addition of a constant) the *potential* for u .

The motivation for this restriction is quite compelling: if we can reduce our system to studying the evolution of the scalar function ϕ alone, then the Krylov-Safonov theory should supply us with all the regularity we need. The example to have in mind is the flow (1.0.1): if $u^\alpha = \partial_\alpha \phi$, then $Du = D^2\phi$ is symmetric and thus its singular values are just its eigenvalues. Thus we see that u satisfies $\partial_t u = (\sigma_1 + \sigma_2)^{-2} \Delta u$ if ϕ satisfies the nonlinear parabolic equation

$$\partial_t \phi = -\frac{1}{\Delta \phi}.$$

This correspondence is one of the primary reasons that (1.0.1) is the right flow for our job. We now investigate whether or not there are any other flows with this same kind of correspondence.

In order for this approach to make sense, we need the flow to preserve the fact that u is exact, so that the definition $u^\alpha = \partial_\alpha \phi$ makes sense on the full space-time domain; and we also need ϕ to satisfy some nice parabolic PDE so that the theory is applicable. Thus, we need to look for flows of ϕ of the form

$$\partial_t \phi = f(D\phi, D^2\phi) \tag{5.1.1}$$

preserving the fact that $\nabla \phi$ is a diffeomorphism, where $f(v, \xi)$ is elliptic in the nonlinear sense; i.e. the matrix $M_{\alpha i}$ of derivatives $\partial f / \partial \xi_{\alpha i}$ is positive-definite. (The lack of dependence on ϕ here is quite necessary - since $u = \nabla \phi$ is blind to transformations $\phi \mapsto \phi + C$, the evolution of ϕ had best not depend on constants in any nonlinear fashion.) Differentiating the flow equation in the direction ∂_α yields the corresponding equation for u :

Proposition 5.1.1. *If $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies the evolution equation $\partial_t \phi = f(D\phi, D^2\phi)$*

then the gradient $u = \nabla \phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies

$$\partial_t u^\alpha = \frac{\partial f}{\partial \xi_{ij}}(u, Du) u_{ij}^\alpha + \frac{\partial f}{\partial v^i}(u, Du) u_i^\alpha.$$

Note that this is always a quasilinear evolution equation - if we impose the isometry invariance condition then it fits in to the framework we set up in Chapter 3. Since the singular values are the eigenvalues, requiring isometry invariance imposes the constraint that

$$\frac{\partial f}{\partial \xi_{ij}}(\xi) = \sum_k F_k(\lambda) v_k^i v_k^j$$

where $\{\lambda_k, v_k\}$ are the eigenvalues and eigenvectors of ξ . Any flow (in the sense of Chapter 3) that can be written in this form and that has a maximum principle providing bounds above and below on the singular values will at least be well behaved for exact initial data: the evolution equation (5.1.1) for the potential will then be a uniformly parabolic scalar PDE, so the Krylov-Safonov theory (along with a Schauder bootstrap) tells us that all derivatives of ϕ (and thus u) will be controlled.

Remark. While we are discussing only the case of \mathbb{R}^n to keep things simple, all of this really works on quotients of \mathbb{R}^n . For example, we want u to be a lift of a diffeomorphism of the torus $\mathbb{R}^n/\mathbb{Z}^n$, it must be of the form $u(x) = Tx + v(x)$ where $T \in SL(n, \mathbb{Z})$ and v is \mathbb{Z}^n -periodic. In the case where T is symmetric we can arrange this by letting $\phi(x) = \frac{1}{2}x^T T x + \psi(x)$ for some \mathbb{Z}^2 -periodic function ψ .

We now need to see what is necessary in order to extend this regularity to *all* solutions, not just exact ones. We will start by investigating the structure of the coefficients of these potential flows.

For the rest of this section, let $a^{ij} = \partial f / \partial \xi_{ij}$ be the coefficients of an isometry-invariant flow derived from a potential flow of the form (5.1.1). We first investigate the constraints that coming from a potential flow imposes on the defining function $a(\lambda_1; \text{other } \lambda)$ in Proposition 3.4.11. To simplify the notation, we will instead write our flow coefficients as $a^{ij} = \tilde{a}^{kl}(\lambda_1, \dots, \lambda_n) e_k^i e_l^j$ with the understanding that \tilde{a} is diagonal and satisfies the equivariance condition. The Poincaré lemma tells us that a^{ij} is a potential flow if and only if the “curl” $X^{ijkl} := \partial a^{ij} / \partial \xi_{kl} - \partial a^{kl} / \partial \xi_{ij}$ vanishes identically on the set $S = \text{Sym}^2 \mathbb{R}^n$ of symmetric matrices.

In the unexceptional region $\mathcal{U} \subset S$ where the eigenvalues are distinct, we can use the eigenvalues and vectors as coordinates for S ; and in fact proving the curl is zero in this regime suffices to prove it is zero everywhere, since this region is dense in S . Applying equation (3.4.8) in this special setting yields the formula

$$\frac{\partial a^{ij}}{\partial \xi_{kl}} = \sum_{a,c} \frac{\partial F_c}{\partial \lambda_a} e_c^i e_c^j e_a^k e_a^l + 4 \sum_{a < b} \frac{F_a - F_b}{\lambda_a - \lambda_b} e_b^{(i} e_a^{j)} e_a^{(k} e_b^{l)}.$$

The second term is manifestly symmetric under the interchange $ij \leftrightarrow kl$, so the condition

$X^{ijkl} = 0$ reduces to

$$\frac{\partial F_c}{\partial \lambda_a} + \frac{\partial F_a}{\partial \lambda_c} = 0;$$

i.e. to the condition that the equivariant speed function $F : (0, \infty)^n \rightarrow (0, \infty)^n$ is conservative when interpreted as a vector field on $(0, \infty)^n$.

A natural thing to investigate now is the relationship between the potentials for a^{ij} and F : if $F_a = \partial_a \varphi$ for some $\varphi : (0, \infty)^n \rightarrow \mathbb{R}$, what is the relationship between φ and f ? The natural guess is $f(\xi) = \varphi(\lambda_1(\xi), \dots, \lambda_n(\xi))$, and it turns out this is correct: we have

$$\frac{\partial f}{\partial \xi_{kl}} = \frac{\partial \varphi}{\partial \lambda_a} \frac{\partial \lambda_a}{\partial \xi_{kl}} = \sum_a F_a e_a^k e_a^l = a^{kl}$$

as desired. Finally, note that the equivariance condition (swapping λ_a and λ_b swaps F_a and F_b) can be achieved simply by requiring φ to be invariant under the natural action of the symmetric group S_n on $(0, \infty)^n$. Thus we have found a class of flows that at least have local Hölder estimates for exact initial data:

Definition 5.1.2. An *invariant potential* map flow is an isometry-invariant geometric map flow specified by coefficients

$$a^{ij}(Du) = \sum_a \frac{\partial \varphi}{\partial \sigma_a} e_a^i e_a^j$$

for some smooth symmetric function $\varphi : (0, \infty)^n \rightarrow \mathbb{R}$ of the singular values.

Note that the condition that the data is exact is locally equivalent to Du being symmetric; so for this class of data the SVD $Du = V \Sigma E^T$ is in fact an orthogonal diagonalization $Du = E \Sigma E^T$. We can extend to a slightly larger class of data by applying a constant rotation: If we let $\Theta \in O(n)$ be fixed and consider $\tilde{u} = \Theta u$ for a exact solution u of an invariant potential flow, the isometry invariance of f implies that \tilde{u} is also a solution; and clearly has just as good regularity. (This should not be surprising - all we're doing is rotating our coordinate axes!) A solution of this form will instead have an SVD that looks like $Du = V \Sigma E^T$ with $V = \Theta E$; so $\Theta = V E^T$ is the rotational component of the right polar decomposition $Du = \Theta S = (V E^T)(E \Sigma E^T)$. Thus we can describe these “good” diffeomorphisms in a new way - they are exactly those whose derivatives have constant rotational component. This suggests a method of attack for regularity in the general case: since invariant potential flows preserve the fact that Θ is constant, we can hope that, for general data, Θ at least satisfies a nice evolution equation. Recall from Proposition 3.4.9 that Θ is a smooth function of Du .

5.2 Evolution of the Rotation Angle

Let's now restrict ourselves to orientation-preserving maps and $n = 2$, so that the rotational component $\Theta \in SO(2)$ of the derivative is just rotation by some angle θ . Remember that for invertible Du , the polar decomposition is unique.

Proposition 5.2.1. *If $Du = \Theta S$ is the polar decomposition of a 2×2 matrix $Du = (u_i^\alpha)$ with $\det Du > 0$, then Θ is a rotation by the angle*

$$\theta = \text{atan2}(u_1^2 - u_2^1, u_1^1 + u_2^2);$$

i.e. the angle between the positive x -axis and the point $(u_1^1 + u_2^2, u_1^2 - u_2^1)$, measured counterclockwise from the origin.

Proof. Define $\xi = u_1^1 + u_2^2$ and $\zeta = u_1^2 - u_2^1$. Since

$$S = \Theta^T Du = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} u_1^1 & u_2^1 \\ u_1^2 & u_2^2 \end{pmatrix}$$

is symmetric, we must have equality of the off-diagonal terms; i.e.

$$u_2^1 \cos \theta + u_2^2 \sin \theta = -u_1^1 \sin \theta + u_1^2 \cos \theta.$$

Manipulating this yields $\zeta \cos \theta = \xi \sin \theta$ and thus $\tan \theta = \zeta/\xi$. Finally, the fact that S is positive-definite tells us that the trace of $\Theta^T Du$ is positive, which yields $\xi \cos \theta + \zeta \sin \theta > 0$; so (ξ, ζ) cannot be opposite $(\cos \theta, \sin \theta)$ and thus must be in the same direction. \square

Thus we first investigate the evolution of the quantities ξ and ζ . We start with equation (4.2.2), for now working in Cartesian coordinates in flat space. Since we are interested in determining the evolution of θ and we know how θ transforms under rigid motions of \mathbb{R}^2 , we can choose to do this calculation in the SVD frame at a single point - so long as the equation we arrive at depends on θ only through its derivatives (which are invariant under rotations) and on u only through its isometry invariants, it will in fact be true everywhere in arbitrary Cartesian coordinates. Starting from $Pu_k^\alpha = \frac{\partial a^{ij}}{\partial x^k} u_{ij}^\alpha$ for a^{ij} the invariant flow associated to a potential φ , we substitute 3.4.11 and arrive at

$$Pu_k^\alpha = \sum_{i,j} \ddot{\varphi}^{ij} u_{jk}^j u_{ii}^\alpha + 2 \frac{(\dot{\varphi}^1 - \dot{\varphi}^2)}{\sigma_1^2 - \sigma_2^2} (\sigma_1 u_{2k}^1 + \sigma_2 u_{1k}^2) u_{12}^\alpha, \quad (5.2.1)$$

from which we can find

$$\begin{aligned} P\xi &= Pu_1^1 + Pu_2^2 \\ &= \ddot{\varphi}^{ij} u_{jk}^j u_{ii}^k + 2 \frac{\dot{\varphi}^1 - \dot{\varphi}^2}{\sigma_1^2 - \sigma_2^2} (\sigma_1 (u_{12}^1 u_{12}^1 + u_{22}^1 u_{12}^2) + \sigma_2 (u_{11}^2 u_{12}^1 + u_{12}^2 u_{12}^2)) \end{aligned}$$

and

$$\begin{aligned} P\zeta &= Pu_1^2 - Pu_2^1 \\ &= \ddot{\varphi}^{ij} (u_{j1}^j u_{ii}^2 - u_{j2}^j u_{ii}^1) + 2 \frac{\dot{\varphi}^1 - \dot{\varphi}^2}{\sigma_1^2 - \sigma_2^2} (\sigma_1 (u_{21}^1 u_{12}^2 - u_{22}^1 u_{12}^1) + \sigma_2 (u_{11}^2 u_{12}^2 - u_{12}^2 u_{12}^1)). \end{aligned}$$

Now, since the formula for θ in terms of ξ and ζ we found implies $\xi = r \cos \theta, \zeta = r \sin \theta$ for $r = \sqrt{\xi^2 + \zeta^2}$, we can differentiate these relations and take linear combinations to get the following expressions for the derivatives of θ :

$$\begin{aligned} r^2 d\theta &= \xi d\zeta - \zeta d\xi \\ r^2 D^2\theta &= \xi D^2\zeta - \zeta D^2\xi - 2r dr d\theta. \end{aligned}$$

Combining these we get the evolution equation for θ in terms of those for ξ, ζ :

$$P\theta = \frac{1}{r^2} (\xi P\zeta - \zeta P\xi) + \frac{2}{r} a(dr, d\theta).$$

Substituting the evolution equations for ξ, ζ yields (noting $\zeta = 0, \xi = r = \sigma_1 + \sigma_2$ at the origin of the SVD coordinates)

$$P\theta = \frac{1}{r} \dot{\varphi}^{ij} (u_{j1}^j u_{ii}^2 - u_{j2}^j u_{ii}^1) + \frac{2}{r^2} \frac{\dot{\varphi}^1 - \dot{\varphi}^2}{\sigma_1^2 - \sigma_2^2} (\sigma_1 u_{12}^1 \partial_2 \zeta + \sigma_2 u_{12}^2 \partial_1 \zeta) + \frac{2}{r} \dot{\varphi}^a \partial_a r \partial_a \theta.$$

Now, in order for this to tell us anything about the regularity of θ , we need to be able to rewrite the $D^2 u$ terms in order to reduce the size of the system we're dealing with. At the very least we should try to get things in terms of θ, r alone, so we make the assumption $\varphi(\sigma_1, \sigma_2) = \psi(\sigma_1 + \sigma_2)$ to eliminate the $\dot{\varphi}^1 - \dot{\varphi}^2$ term (since e.g. $u_{12}^1 \partial_2 \zeta$ cannot be written in terms of r and θ), which yields (noting that $dr = d\xi, r d\theta = d\zeta$ at the origin and that we can write u_{ii}^k in terms of derivatives of ξ, ζ)

$$P\theta = \left(\psi'' + \frac{2\psi'}{r} \right) \langle dr, d\theta \rangle.$$

This additional assumption can be rephrased as asserting that the principal symbol of P is proportional to the (inverse) metric; i.e. that the diffusion is isotropic.

We see now that $P\theta = 0$ if and only if $\psi'(r) \propto r^{-2}$; i.e. the flow is a multiple of (1.0.1). Otherwise, we get a non-zero $\langle dr, d\theta \rangle$ term, so we have to treat r, θ as a system of two equations. Since the regularity of the flow coefficients $a^{ij} = \psi'(r) g^{ij}$ is now determined entirely by the regularity of r , this is promising - mild assumptions on ψ and $C^{0,\alpha}$ estimates for this system will imply smoothness for the full flow. Thus we do very similar calculations to those above, this time for the evolution of r :

$$\begin{aligned} r dr &= \xi d\xi + \zeta d\zeta \\ dr^2 + r D^2 r &= \xi D^2 \xi + \zeta D^2 \zeta + d\xi^2 + d\zeta^2 \\ Pr &= \frac{1}{r} (\xi P\xi + \zeta P\zeta) - r a(d\theta, d\theta) \end{aligned}$$

These along with the same assumptions we used above yield

$$\begin{aligned} Pr &= \psi''(r) \left(|d\xi|^2 + d\xi \times d\zeta \right) - \frac{1}{r} a(d\zeta, d\zeta) \\ &= \psi'' |dr|^2 + r\psi'' dr \times d\theta - r\psi' |d\theta|^2 \end{aligned}$$

where $a \times b$ denotes the two-dimensional cross product of one-forms $\star(a \wedge b) = a_2 b_1 - a_1 b_2$. Thus we have a coupled system

$$\begin{aligned} \partial_t \theta &= \psi'(r) \Delta \theta + \left(\psi''(r) + \frac{2\psi'(r)}{r} \right) \langle dr, d\theta \rangle \\ \partial_t r &= \psi'(r) \Delta r + \psi''(r) \left(|dr|^2 + r dr \times d\theta \right) - r\psi'(r) |d\theta|^2 \end{aligned}$$

of two quasilinear PDE in 2+1 variables, with nonlinearity determined by a smooth increasing function $\psi : (0, \infty) \rightarrow \mathbb{R}$. Note also that these equations are rotationally invariant, so (as discussed when we made the simplifying choice of coordinates) they in fact hold everywhere. The presence of the gradient reaction terms here is quite problematic - if we're looking for $C^{0,\alpha}$ estimates then we have at best C^0 control already, so there's not much we can do. Thus (in keeping with our idea of finding a flow that treats θ nicely) we restrict ourselves to the solution of $\psi''(r) + 2\psi'(r)/r = 0$, which is (fixing the constant of integration with a sign making a^{ij} elliptic) $\psi(r) = -1/r$. This is the potential that gives rise to the flow (1.0.1). This reduces the system to

$$\partial_t \theta = \frac{1}{r^2} \Delta \theta \tag{5.2.2}$$

$$\partial_t r = \frac{1}{r^2} \Delta r - \frac{2}{r^3} \left(|dr|^2 + r dr \times d\theta \right) - \frac{1}{r} |d\theta|^2, \tag{5.2.3}$$

which achieves our goal of making θ solve a uniformly parabolic equation, and thus delivers us a Hölder estimate for θ . We will see in the next section how we can use this estimate along with (5.2.3) to get the corresponding estimate for r .

5.3 Hölder Estimates for the Coefficient

In the previous section we restricted ourselves to flat space for clarity of the main idea; but we will see now that the same approach works in the Riemannian setting if we interpret θ in the correct way. Thus we will spend the first part of this section deriving a Riemannian analogue of the Euclidean system (5.2.2), (5.2.3). For simplicity, we will assume M, N are compact oriented Riemannian surfaces - in particular, they both have a canonical complex structure given by the unique positively-oriented rotation of tangent spaces by $\pi/2$. Since we have already decided to focus on the flow (1.0.1) defined by

$$\varphi(\sigma_1, \sigma_2) = \psi(\sigma_1 + \sigma_2) = -r^{-1} = -(\sigma_1 + \sigma_2)^{-1},$$

the evolution equation for the derivative is the quite manageable

$$(PDu)_k^\alpha = \nabla_k F \Delta u^\alpha - F g^{ij} R_{ikj\beta}^{\alpha l} \nabla_l u^\beta$$

where $F = r^{-2}$. Writing $\nabla_k F = g_{ij} \nabla_k a^{ij}$ and applying Proposition 3.4.19, we have (still using abstract indices)

$$\nabla_k F = -\frac{2}{r^3} \sum_a e_a^i v_a^\beta g_{\alpha\beta} \nabla_k \nabla_i u^\alpha = -\frac{2}{r^3} (\Theta^{-1})_\alpha^i \nabla_k \nabla_i u^\alpha \quad (5.3.1)$$

and thus

$$(PDu)_k^\alpha = -\frac{1}{r^2} \left(\frac{2}{r} (\Theta^{-1})_\beta^i \nabla_k \nabla_i u^\beta \Delta u^\alpha + g^{ij} R_{ikj\beta}^{\alpha l} \nabla_l u^\beta \right), \quad (5.3.2)$$

where Θ is the rotational component of Du as defined in Proposition 3.4.10.

Fix a point $X_0 \in M \times (0, T)$ and a small radius $R_0 > 0$, and let $E_a \in \Gamma_{\text{loc}}(FO\mathfrak{S})$, $V_a \in \Gamma_{\text{loc}}(FOu^*TN)$ be orthonormal frames defined on $Q(X_0, R_0)$ agreeing with the singular frames at x_0 . Define θ to be the angle of rotation of Θ with respect to these frames; i.e. the unique function vanishing at X_0 such that $V^{-1}\Theta E = \exp(i\theta)$ where $i \in \mathfrak{so}(2)$ is the counter-clockwise rotation by $\pi/2$.¹ This can be described in terms of principal bundles: the orthogonality of Θ implies that it is a section of the $SO(2)$ -bundle $SO(\mathfrak{S}, u^*TN)$. By choosing E, V we are in some sense fixing a gauge for this bundle, so that the section Θ can be studied via its local coordinate representative $i\theta : Q(X_0, R_0) \rightarrow \mathfrak{so}(2)$.

Proposition 5.3.1. *If $u : M \times [0, T) \rightarrow N$ is a solution of the flow $\partial_t u = r^{-2} \Delta u$ satisfying gradient bounds $\lambda_- \leq \sigma(Du) \leq \lambda_+$, then the functions θ and r satisfy the 2×2 system of parabolic PDE*

$$P\theta = \langle PE_1, E_2 \rangle - \langle PV_1, V_2 \rangle + f_\theta \quad (5.3.3)$$

$$Pr = -\frac{2}{r^3} |dr|^2 - \frac{2}{r^2} dr \times d\theta - \frac{1}{r} |d\theta|^2 - \langle b_r, dr \rangle - \langle b_\theta, d\theta \rangle - f_r. \quad (5.3.4)$$

where the error terms $f_\theta, f_r, |b_r|, |b_\theta|$ are bounded in terms of the connection forms $\langle \nabla E_1, E_2 \rangle, \langle \nabla V_1, V_2 \rangle$ along with $\lambda_\pm, |R^M|, |R^N|, |\nabla R^M|, |\nabla R^N|, \iota_M, \iota_N$. Here P denotes $\nabla_t - r^{-2} \text{tr}_g \nabla^2$ for the natural connection ∇ on each of the bundles \mathfrak{S}, u^*TN .

Proof. As the polar decomposition of Du depends only upon the inner products on the relevant tangent spaces, the formulae we found for ξ, ζ, r can be transferred directly from the Euclidean case so long as we use orthonormal frames: if we define

$$\begin{aligned} \xi &= \langle Du(E_1), V_1 \rangle + \langle Du(E_2), V_2 \rangle \\ \zeta &= \langle Du(E_1), V_2 \rangle - \langle Du(E_2), V_1 \rangle \end{aligned}$$

¹The name i is intended to be conflated with the imaginary unit: if we identify $SO(2)$ with $U(1)$, then i really is the imaginary unit under the corresponding identification of Lie algebras $\mathfrak{so}(2) = \mathfrak{u}(1) = i\mathbb{R}$.

then we have

$$r = \sqrt{\xi^2 + \zeta^2}$$

$$\Theta = \frac{\xi}{r} \left(V_1 \otimes E_1^b + V_2 \otimes E_2^b \right) + \frac{\zeta}{r} \left(V_2 \otimes E_1^b - V_1 \otimes E_2^b \right).$$

Thus from our definition of θ we see that

$$r \exp(i\theta) = \begin{pmatrix} \xi & -\zeta \\ \zeta & \xi \end{pmatrix};$$

i.e. r, θ are the polar coordinates corresponding to ξ, ζ , just as in the previous section. Thus the formulae

$$P\theta = \frac{1}{r^2} (\xi P\zeta - \zeta P\xi) + \frac{2}{r} a(dr, d\theta) \quad (5.3.5)$$

$$Pr = \frac{1}{r} (\xi P\xi + \zeta P\zeta) - ra(d\theta, d\theta) \quad (5.3.6)$$

still hold, and we have

$$(\Theta^{-1})_{\beta}^i \nabla \cdot \nabla_i u^{\beta} = \frac{\xi}{r} \sum_a \nabla^2 u(\cdot, E_a, V_a) + \frac{\zeta}{r} (\nabla^2 u(\cdot, E_2, V_1) - \nabla^2 u(\cdot, V_1, E_2)).$$

The calculations going forward will be quite dense, so for brevity we will switch to index notation in the frames E, V ; e.g. $\nabla_i \nabla_j u^k = (\nabla^2 u)(E_i, E_j, V_k)$. We will write $\omega_k = \omega_{1k}^2 = \langle \nabla_{E_k} E_1, E_2 \rangle$ and $\tau_k = \tau_{1k}^2 = \langle \nabla_{E_k} V_1, V_2 \rangle$ for the determining components of the connection forms². Since we want to get a closed system for ξ, ζ (and thus for θ, r), we will need expressions for $d\xi, d\zeta$ in these frames. The product rule for the natural connections yields

$$\partial_k \xi = \nabla_k \nabla_a u^a + (\tau_k - \omega_k) (\nabla_1 u^2 - \nabla_2 u^1) = \nabla_k \nabla_a u^a - \zeta (\omega_k - \tau_k) \quad (5.3.7)$$

and

$$\begin{aligned} \partial_k \zeta &= \nabla_k \nabla_1 u^2 - \nabla_k \nabla_2 u^1 + (\omega_k - \tau_k) (\nabla_2 u^2 + \nabla_1 u^1) \\ &= \nabla_k \nabla_1 u^2 - \nabla_k \nabla_2 u^1 + \xi (\omega_k - \tau_k). \end{aligned} \quad (5.3.8)$$

The term $\omega_k - \tau_k$ will be coming up a lot; so we define $\eta = \omega - \tau$. Combining the equations above we find

$$\begin{aligned} (\Theta^{-1})_{\beta}^i \nabla \cdot \nabla_i u^{\beta} &= \frac{\xi}{r} (d\xi + \zeta \eta) + \frac{\zeta}{r} (d\zeta - \xi \eta) \\ &= \frac{\xi}{r} d\xi + \frac{\zeta}{r} d\zeta = dr, \end{aligned}$$

²Note that these are *not* the same ω, τ we found formulae for earlier in this thesis, since they are associated to the smooth frames E, V rather than the (possibly non-smooth) singular frames.

as in the Euclidean case; so as we expect the evolution equation for Du reduces to

$$(PDu)_k^\alpha = -\frac{2}{r^3} \partial_k r \Delta u^\alpha + g^{ij} R_{ikj\beta}^{\alpha l} \nabla_l u^\beta.$$

For the Laplacian in the Euclidean case we had $\Delta u^1 = \partial_1 \xi - \partial_2 \zeta$, $\Delta u^2 = \partial_2 \xi + \partial_1 \zeta$, so expanding the RHS of these equations using (5.3.7), (5.3.8) and using the symmetry of $\nabla^2 u$ we find

$$\begin{aligned} \partial_1 \xi - \partial_2 \zeta &= \Delta u^1 - \zeta \eta_1 - \xi \eta_2 \\ \partial_2 \xi + \partial_1 \zeta &= \Delta u^2 - \zeta \eta_2 + \xi \eta_1. \end{aligned}$$

Thus we have

$$\begin{aligned} (PDu)_1^1 &= -\frac{2}{r^3} \partial_1 r (\partial_1 \xi - \partial_2 \zeta + \zeta \eta_1 + \xi \eta_2) + \Omega_1^1 \\ (PDu)_2^2 &= -\frac{2}{r^3} \partial_2 r (\partial_2 \xi + \partial_1 \zeta + \zeta \eta_2 - \xi \eta_1) + \Omega_2^2 \\ (PDu)_1^2 &= -\frac{2}{r^3} \partial_1 r (\partial_2 \xi + \partial_1 \zeta + \zeta \eta_2 - \xi \eta_1) + \Omega_1^2 \\ (PDu)_2^1 &= -\frac{2}{r^3} \partial_2 r (\partial_1 \xi - \partial_2 \zeta + \zeta \eta_1 + \xi \eta_2) + \Omega_2^1 \end{aligned}$$

where $\Omega_k^\alpha = -r^{-2} R_{ikj\beta}^{\alpha l} \nabla_l u^\beta$ can be bounded in terms of λ_\pm and the curvature. Now, to express $P\xi, P\zeta$ in terms of PDu , we spend some more quality time with the product rule, eventually arriving at

$$\begin{aligned} \Delta \xi &= (\Delta Du)_1^1 + (\Delta Du)_2^2 + \xi |\eta|^2 - 2 \langle \eta, d\zeta \rangle - \delta \eta \zeta \\ \Delta \zeta &= (\Delta Du)_1^2 - (\Delta Du)_2^1 + \zeta |\eta|^2 + 2 \langle \eta, d\xi \rangle + \delta \eta \xi \end{aligned}$$

where $\delta \eta = \langle \Delta E_1, E_2 \rangle - \langle \Delta V_1, V_2 \rangle$ is the spatial codifferential of η . Similarly we have

$$\begin{aligned} \nabla_t \xi &= (\nabla_t Du)_1^1 + (\nabla_t Du)_2^2 - \eta_t \zeta \\ \nabla_t \zeta &= (\nabla_t Du)_1^2 + (\nabla_t Du)_2^1 + \eta_t \xi; \end{aligned}$$

so combining all of this we find

$$\begin{aligned} P\xi &= -\frac{2}{r^3} \left(\langle dr, d\xi \rangle - dr \times d\zeta + \zeta \langle \eta, dr \rangle - \xi \eta \times dr + \frac{1}{2} r \xi |\eta|^2 - r \langle \eta, d\zeta \rangle + \frac{1}{2} (\eta_t - r^2 \delta \eta) r \zeta \right) + \Omega_1^1 + \Omega_2^2 \\ P\zeta &= -\frac{2}{r^3} \left(\langle dr, d\zeta \rangle + dr \times d\xi - \xi \langle \eta, dr \rangle - \zeta \eta \times dr + \frac{1}{2} r \zeta |\eta|^2 + r \langle \eta, d\xi \rangle - \frac{1}{2} (\eta_t - r^2 \delta \eta) r \xi \right) + \Omega_1^2 - \Omega_2^1 \end{aligned}$$

where $\alpha \times \beta = \alpha_1 \beta_2 - \alpha_2 \beta_1 = \star(\alpha \wedge \beta)$ denotes the two-dimensional scalar cross product.

Substituting these expressions in to (5.3.5), (5.3.6) we find

$$P\theta = \eta_t - r^2 \delta \eta + \frac{\xi}{r^2} (\Omega_1^2 - \Omega_2^1) - \frac{\zeta}{r^2} (\Omega_1^1 + \Omega_2^2) \quad (5.3.9)$$

and

$$\begin{aligned} Pr = & -\frac{2}{r^3} |dr|^2 + \frac{2}{r^2} dr \times d\theta - \frac{1}{r} |d\theta|^2 \\ & + \frac{2}{r^2} \eta \times dr - \frac{1}{r} |\eta|^2 - \frac{2}{r} \langle \eta, d\theta \rangle + \frac{\xi}{r} (\Omega_1^1 + \Omega_2^2) + \frac{\zeta}{r} (\Omega_1^2 - \Omega_2^1), \end{aligned}$$

which can be estimated as described. \square

In order for these evolution equations to be of any use to us, we need to choose frames in which all the error terms can be bounded by our controlled quantities. For the frame E_a of the tangent bundle of the domain, this choice is easy: just take the singular frames at X_0 , parallel transport them along radial geodesics to cover $B(X_0, R_0)$ and declare them to be constant in time. For V_a , the time-dependence of the bundle u^*TN (or rather its geometry) makes a similar approach untenable: if we parallel transport first in space and then in time, the temporal curvature of u^*TN will introduce terms involving $\partial_t u$ in to the connection forms of V_a , which we have no control over.

The approach that works is one that exploits the evolution equation for u : let W_a be an orthonormal frame for TN constructed by parallel transport along radial geodesics from $u(X_0)$, and then define $V_a \in \Gamma_{\text{loc}}(u^*TN)$ by restriction: $V_a = (W_a)_u$. In order for this to make sense we need u to stay inside the domain of V_a . Since we know u has maximum “speed” λ_+ , we can do this by choosing an R_0 in a manner that depends only on initial data and geometry:

Lemma 5.3.2. *Suppose that $R_0 < \min(\iota_M, \lambda_+^{-1} \iota_N)$ and the frames E_a, V_a are defined on $Q(X_0, R_0)$ as described above. Then $\langle PE_1, E_2 \rangle$, $\langle PV_1, V_2 \rangle$, $\langle \nabla E_1, E_2 \rangle$ and $\langle \nabla V_1, V_2 \rangle$ can be bounded solely in terms of R_0 , singular value bounds and the geometry of M, N .*

Proof. By construction we know that $\nabla E_1, \Delta E_1$ vanish at X_0 ; the latter because ΔE_1 can be written as the sum of second covariant derivatives in various directions, each of which is radial when evaluated at the origin. To estimate how these quantities change as we move away from the origin it suffices to get a bound on their radial derivatives. Commuting these derivatives we get (along a radial coordinate line of increasing x^i)

$$\begin{aligned} \nabla_i \nabla E_1 &= \nabla \nabla_i E_1 + R(\partial_i, \cdot) E_1 \\ \nabla_i \Delta E_1 &= \Delta \nabla_i E_1 + \sum_j [R(\partial_i, E_j) (\nabla_{E_j} E_1) + \nabla_{E_j} (R(\partial_i, E_j) E_1)]. \end{aligned}$$

Since the first terms in each of these expression vanish by construction, integrating along the radius gives $|\nabla E_1| \leq |R^M| R_0$, and thus in turn $|\Delta E| \leq 2(|R^M| R_0)^2 + |\nabla R^M| R_0$; so since we defined E to be constant in time we have the claimed bounds for the derivatives of E . In very similar fashion we can bound $|\nabla W_1|$ and $|\nabla^2 W_1|$ in terms of the distance from $u(X_0)$, $|R^N|$ and $|\nabla R^M|$. By differentiating the definition $V_1 = (W_1)_u$ using the definition of the pullback connection, we find that the flow equation $\partial_t u = F \Delta u$ implies

$\nabla_i V_1 = \nabla_i u^\alpha (\nabla_\alpha W_1)_u$ along with the nice evolution equation

$$PV_1 = -Fg^{ij}\nabla_i u^\alpha \nabla_j u^\beta (\nabla_\alpha \nabla_\beta W_1)_u, \quad (5.3.10)$$

so the singular value bounds and our control on W are all we need to control PV and derivatives of V . \square

Thus the RHS of (5.3.3) is bounded in terms of λ_\pm, R_0 and geometry; so we immediately obtain the Hölder regularity of θ at X_0 by applying the local Hölder estimate Proposition 2.4.21:

Corollary 5.3.3. *If u satisfies the hypotheses of Proposition 5.3.1, then θ defined using the frames above satisfies $\text{osc}_{Q(X_0, R)} \theta \leq CR^\alpha$ for all $R \in [0, R_0]$ where the constant C and exponent α depend only on $\text{osc}_{Q(X_0, R_0)} \theta$, λ_\pm , R_0 and geometry.*

We can now exploit this regularity of θ along with (5.3.5) to obtain a Hölder estimate for r . In the Euclidean case this estimate has previously appeared in a preprint of the author and Ben Andrews [AC16]. The proof will follow the same general idea as Proposition 2.4.21, though it will need significant modification in order to handle the terms involving $d\theta$. The idea is to study the functions $r^p \pm c\theta^2$ for some small $p > 0$ and carefully chosen $c > 0$: by perturbing r with a little θ , we can hope to control the sign of the resulting reaction term enough to get the subsolution and supersolution that we need. The fact that θ is Hölder will allow us to “invert” the perturbation in the sense that we can recover a Hölder estimate for r alone. The powers p are just around to give us enough freedom to make this work - since we have uniform bounds $\lambda_+^{-2} \leq r \leq \lambda_-^{-2}$, all the positive powers of r are comparable, so there is no real loss here. The idea of perturbing the quantity of interest by something better behaved is not new: see for example the Hölder gradient estimate for the solutions of (scalar) quasilinear parabolic equations in which $\partial_i u$ is perturbed by $|du|^2$ [Lie96, XII.3], which was used at least as early as [LU63, §2].

Let L denote the linear divergence-form operator $Lf = \partial_t f - \text{div}(r^{-2}Df)$.

Lemma 5.3.4. *For θ, r obtained from a flow solution u as in the prequel, there are constants $p \in (0, 1)$, $c > 0$, H, R_0 depending only upon λ_\pm and geometry such that the functions r^3 and $w = r^p - c\theta^2$ satisfy $Lr^3 \leq H$, $Lw \geq -H$ on $Q(4R_0)$.*

Proof. Since we have chosen frames controlling PE, PV in terms of λ_\pm and geometry, we can roll these terms in to f_θ to slightly simplify the quagmire of computation that is to come. Starting with the evolution equations (5.3.3), (5.3.4) we compute

$$\begin{aligned} Pw &= pr^{p-1}Pr - p(p-1)r^{p-4}|dr|^2 - 2c(\theta P\theta - r^{-2}|d\theta|^2) \\ &= -p(p+1)r^{p-4}|dr|^2 + 2pr^{p-3}dr \times d\theta - r^{-2}(pr^p - 2c)|d\theta|^2 \\ &\quad - pr^{p-1}(\langle b_r, dr \rangle + \langle b_\theta, d\theta \rangle + f_r) - 2c\theta f_\theta, \end{aligned}$$

and thus when we switch to the divergence-form operator (noting $Lf = Pf + 2r^{-3} \langle dr, df \rangle$) we get

$$Lw = p(1-p)r^{p-4}|dr|^2 + 2pr^{p-3}dr \times d\theta - r^{-2}(pr^p - 2c)|d\theta|^2 - pr^{p-1}(\langle b_r, dr \rangle + \langle b_\theta, d\theta \rangle + f_r) - 2c\theta f_\theta + 4c\theta r^{-3} \langle dr, d\theta \rangle. \quad (5.3.11)$$

Setting $p = 3$, $c = 0$ we find

$$Lr^3 = -6r^{-1}|dr|^2 + 6dr \times d\theta - 3r|d\theta|^2 - 3r^2(\langle b_r, dr \rangle + \langle b_\theta, d\theta \rangle + f_r).$$

Using Peter-Paul we can estimate $6|dr \times d\theta| \leq 6|dr||d\theta| \leq \frac{9}{2}r^{-1}|dr|^2 + 2r|d\theta|^2$, yielding³

$$Lr^3 \leq -\frac{3}{2}r^{-1}|dr|^2 - r|d\theta|^2 - 3r^2(\langle b_r, dr \rangle + \langle b_\theta, d\theta \rangle + f_r).$$

Applying Peter-Paul again in the form $|3r^2 \langle b_r, dr \rangle| \leq \frac{3}{2}r^{-1}|dr|^2 + \frac{3}{2}r^3|b_r|^2$ and similarly for b_θ , we find

$$Lr^3 \leq \frac{3}{2}r^3|b_r|^2 + \frac{3}{4}r^3|b_\theta|^2 + 3r^2|f_r|,$$

which is bounded in terms of λ_\pm and geometry. To get a supersolution involves some similar estimates but is a little tougher: we need to include a positive c term to get a good $|d\theta|^2$ term, and we will have to be a little more careful with our choice of p . Start by estimating $|dr \times d\theta|, |\langle dr, d\theta \rangle| \leq \frac{1}{3r}|dr|^2 + \frac{3r}{4}|d\theta|^2$ in (5.3.11), which yields

$$Lw \geq \left[p \left(\frac{1}{3} - p \right) r^{p-4} - \frac{4}{3}c\theta r^{-4} \right] |dr|^2 - \left[c(2 - 3\theta)r^{-2} - \frac{5}{2}pr^{p-2} \right] |d\theta|^2 - pr^{p-1}(\langle b_r, dr \rangle + \langle b_\theta, d\theta \rangle + f_r) - 2c\theta f_\theta.$$

Now estimate $|pr^{p-1} \langle b_r, dr \rangle| \leq \frac{1}{6}pr^{p-4}|dr|^2 + \frac{3}{2}pr^{p+2}|b_r|^2$ and $|pr^{p-1} \langle b_\theta, d\theta \rangle| \leq cr^{-2}|d\theta|^2 + \frac{1}{4c}p^2r^{2p}|b_\theta|^2$, which reduces this to

$$Lw \geq \left[p \left(\frac{1}{6} - p \right) r^{p-4} - \frac{4}{3}c\theta r^{-4} \right] |dr|^2 - \left[c(1 - 3\theta)r^{-2} - \frac{5}{2}pr^{p-2} \right] |d\theta|^2 - \left(\frac{3}{2}pr^{p+2}|b_r|^2 + \frac{1}{4c}p^2r^{2p}|b_\theta|^2 + pr^{p-1}f_r + 2c\theta f_\theta \right). \quad (5.3.12)$$

We want to choose positive p, c to make the two bracketed expressions positive; i.e. such

³Note that if we chose $p = 2$ here (as in the Euclidean case [AC16]) then this Peter-Paul estimate would only barely squeak by, so we wouldn't have any $|dr|^2, |d\theta|^2$ left over to deal with the terms arising from the background geometry.

that

$$\begin{aligned} p \left(\frac{1}{6} - p \right) r^p &\geq \frac{4}{3} c \theta \\ c(1 - 3\theta) &\geq \frac{5}{2} p r^p. \end{aligned}$$

These are both satisfied if $|\theta| < \delta < 1$ and

$$\frac{5}{2} \frac{p \lambda_-^{-2p}}{1 - 3\delta} \leq c \leq \frac{3}{4} \frac{p \left(\frac{1}{6} - p \right)}{\delta} \lambda_+^{-2p},$$

and for such a c to exist we just need

$$\frac{10}{3} \frac{\delta}{1 - 3\delta} \left(\frac{\lambda_+}{\lambda_-} \right)^{2p} \leq \frac{1}{6} - p;$$

which is true for small positive p when δ is small enough. Since we know θ is uniformly Hölder continuous, we can shrink R_0 until $|\theta|$ is bounded by the requisite δ on $Q(4R_0)$, making the coefficients of $|dr|^2$ and $|d\theta|^2$ positive. Since $2c\theta f_\theta \geq -2c\delta |f_\theta|_0$, the remaining terms are bounded by controlled quantities; so noting that δ, p, c and our new R_0 were chosen based only on the desired data, we are done. \square

Now that we have our super- and sub-solutions, we can apply the weak Harnack inequality as in Proposition 2.4.21, but there is one small hitch: our Hölder estimate for θ depends on the local bound $\text{osc}_{Q(X_0, 4R_0)} \theta$, which we do not yet have control of. In PDE the dependence of the Hölder constant on this “large-scale” oscillation is usually not worth a second thought - we just bound it by $2|\theta|_0$ and move on. For the purposes of proving Theorem 1.0.1, we can exploit the Euclidean structure to get the bound we need from the maximum principle:

Proposition 5.3.5. *Suppose M, N are flat 2-dimensional tori and u is a solution of the flow (1.0.1). Then there is a constant C depending only on the initial data such that $\text{osc}_{Q(X_0, 4R_0)} \theta \leq C$ whenever θ is constructed using the nice frames described above.*

Proof. Viewing the tori as quotients of \mathbb{R}^2 by groups of translations, we have fibre-metric-preserving trivializations $TM = M \times \mathbb{R}^2$, $u^*TN = M \times \mathbb{R}^2$ and thus $SO(TM, u^*TN) = M \times SO(2)$; so the rotational component Θ can be viewed as a map $M \times [0, T) \rightarrow SO(2) \simeq \mathbb{R}/2\pi\mathbb{Z}$, where the last identification is via $e^{i\theta} \mapsto \theta \bmod 2\pi$. Lifting this to universal covers we get a function $\tilde{\theta} : \mathbb{R}^2 \times [0, T) \rightarrow \mathbb{R}$. By flatness, the frames E, V defined by parallel transport on $Q(X_0, 4R_0)$ will be constant (when their constituent vectors are viewed as maps $M \rightarrow \mathbb{R}^2$); so the local rotation angle $\theta : Q(X_0, 4R_0) \rightarrow \mathbb{R}$ defined by $e^{i\theta} = V^{-1}\Theta E = V^{-1}e^{i\tilde{\theta}}E$ differs from $\tilde{\theta}$ by addition of some constant c depending only on X_0 . Thus the local PDE $P\theta = 0$ implies $P\tilde{\theta} = P\theta + Pc = 0$ on $Q(X_0, 4R_0)$, so $\tilde{\theta} : \mathbb{R}^2 \times [0, T) \rightarrow \mathbb{R}$ satisfies $P\tilde{\theta} = 0$ everywhere. Decomposing $\tilde{\theta}$ in to the sum of a static linear function T (which is acting as a linear representative of the homotopy class of θ) and a smooth

periodic function ψ , we see that $P\psi = 0$ also; so $\psi : M \times [0, T) \rightarrow \mathbb{R}$ satisfies a maximum principle and is thus bounded in terms of its initial data ψ_0 . Over a ball of radius $4R_0$ we know T can vary by at most $8R_0 |\nabla T|$, so we have $\text{osc}_{Q(X_0, 4R_0)} \theta = \text{osc}_{Q(X_0, 4R_0)} \tilde{\theta} \leq C$ with $C = 8R_0 |\nabla T| + 2|\psi_0|_0$. \square

In the general Riemannian case, the local functions θ do not “glue together” in this way, which makes attaining such a bound more difficult; so for now we will just prove the Hölder estimate under the assumption that we have one. In Appendix C we will remove this assumption by studying the evolution of Θ in a more global manner.

Theorem 5.3.6. *Suppose M, N are compact Riemannian surfaces and $u : [0, T) \rightarrow \text{Diff}(M, N)$ is a solution of the flow (1.0.1) satisfying uniform gradient estimates $\lambda_- \leq \sigma(Du) \leq \lambda_+$, and that we have C^0 control on θ in the sense of Proposition 5.3.5. Then $F(Du) = r^{-2}$ satisfies a uniform Hölder estimate with constant and exponent depending only on $u(0), \lambda_-, \lambda_+$ and geometry.*

Proof. Thanks to Lemma 5.3.4, on any cylinder $Q(4R_0) = Q(X_0, 4R_0)$ we have two supersolutions w and $-r^3$ of the equation $Lu = H$. For any $R < R_0$, let $\Theta(R) = Q((x, t - 4R^2), R)$ be the time-shifted cylinder and apply the divergence-form version of Lemma 2.4.20 to the positive supersolutions $w - \inf_{Q(4R)} w$, $\sup_{Q(4R)} r^3 - r^3$ on the cylinder $Q(4R)$ to get

$$\int_{\Theta(R)} \left(w - \inf_{Q(4R)} w \right) \leq C \left(\inf_{Q(R)} w - \inf_{Q(4R)} w + HR^2 \right) \quad (5.3.13)$$

$$\int_{\Theta(R)} \left(-r^3 + \sup_{Q(4R)} r^3 \right) \leq C \left(-\sup_{Q(R)} r^3 + \sup_{Q(4R)} r^3 + HR^2 \right). \quad (5.3.14)$$

Now, note that for any $q > 0$, our bounds $0 < \lambda_+^{-2} \leq r \leq \lambda_-^{-2} < \infty$ mean that oscillations of the functions r and r^q are uniformly comparable: on the interval $[\lambda_+^{-2}, \lambda_-^{-2}]$ we have bi-Lipschitz control

$$q\Lambda^{-|1-q|} \leq \frac{|s^q - r^q|}{|s - r|} \leq q\Lambda^{|1-q|}$$

where $\Lambda = \max(\lambda_-^{-2}, \lambda_+^2) \in (0, \infty)$. Replacing s by infimums/supremums of r we can thus estimate the integrand in (5.3.13) by

$$-r^3 + \sup_{Q(4R)} r^3 \geq 3\Lambda^{-2} \left(-r + \sup_{Q(4R)} r \right)$$

and similarly that in (5.3.14) by

$$\begin{aligned} w - \inf_{Q(4R)} w &= r^p - \inf_{Q(4R)} r^p + c \left(\sup_{Q(4R)} \theta^2 - \theta^2 \right) \\ &\geq p\Lambda^{p-1} \left(r - \inf_{Q(4R)} r \right), \end{aligned}$$

where we used the fact that $p \in (0, 1)$. Similarly on the right-hand sides we can bound

$$\sup_{Q(4R)} r^3 - \sup_{Q(R)} r^3 \leq 3\Lambda^2 \left(\sup_{Q(4R)} r - \sup_{Q(R)} r \right)$$

and

$$\inf_{Q(R)} w - \inf_{Q(4R)} w \leq p\Lambda^{1-p} \left(\inf_{Q(R)} r - \inf_{Q(4R)} r \right) + c \sup_{Q(4R)} \theta^2.$$

Thus we have massaged the bounds (5.3.13), (5.3.14) into a shape very similar to that of those in Proposition 2.4.21:

$$\begin{aligned} \int_{\Theta(R)} \left(r - \inf_{Q(4R)} r \right) &\leq \frac{1}{p} C \Lambda^{1-p} \left(p\Lambda^{1-p} \left(\inf_{Q(R)} r - \inf_{Q(4R)} r \right) + c \sup_{Q(4R)} \theta^2 + HR^2 \right) \\ \int_{\Theta(R)} \left(-r + \sup_{Q(4R)} r \right) &\leq \frac{1}{3} C \Lambda^2 \left(3\Lambda^2 \left(\sup_{Q(4R)} r - \sup_{Q(R)} r \right) + HR^2 \right). \end{aligned}$$

Consolidating all the constants and then adding these two inequalities, we find

$$\operatorname{osc}_{Q(4R)} r \leq C' \left(\operatorname{osc}_{Q(4R)} r - \operatorname{osc}_{Q(R)} r + \sup_{Q(4R)} \theta^2 + R^2 \right)$$

where $C' = C \max \left(\Lambda^{2-2p}, \frac{c}{p} \Lambda^{1-p}, \Lambda^4, \frac{1}{3} H \Lambda^2, \frac{1}{p} H \Lambda^{1-p} \right)$ depends only on λ_{\pm} and geometry. Rearranging this to

$$\operatorname{osc}_{Q(R)} r \leq \left(1 - \frac{1}{C'} \right) \operatorname{osc}_{Q(4R)} r + \sup_{Q(4R)} \theta^2 + R^2$$

and applying the iteration lemma [GT83, Lemma 8.23] with $\sigma(R) = \sup_{Q(4R)} \theta^2 + R^2$ and $\mu = 1/2$, we obtain

$$\operatorname{osc}_{Q(R)} r \leq C \left(\operatorname{osc}_{Q(R_0)} r \left(\frac{R}{R_0} \right)^{\alpha} + \sup_{Q(4R_0)} |\theta| \sup_{Q(4R)} |\theta| + R_0 R^{\alpha} \right)$$

with C, α controlled; so the Hölder estimate for θ allows us to conclude

$$\operatorname{osc}_{Q(R)} r \leq C \left(\frac{\operatorname{osc}_{Q(R_0)} r}{R_0^{\alpha}} + 4^{\alpha} [\theta]_{\alpha} \sup_{Q(4R_0)} |\theta| + R_0^{2-\alpha} \right) R^{\alpha}$$

as desired. \square

5.4 Higher Derivative Bounds

Once we have a Hölder bound on the coefficients $a^{ij}(x, t)$, we can exploit the Schauder estimate Proposition 2.4.19 and the quasilinear structure of our equation to obtain bounds on all derivatives using a standard bootstrap argument. We use the notation $\nabla^k u \in \Gamma(\otimes^k TM^* \otimes u^* TN)$ to refer to the $(k-1)^{\text{th}}$ iterated covariant derivative of Du , so to avoid confusion we will avoid raising indices on covariant derivatives when using abstract index notation.

We start by deriving a variant of the Schauder estimate for the derivatives of $u : M \times [0, T] \rightarrow N$:

Lemma 5.4.1. *If u is a solution to the flow $\partial_t u = a^{ij}(Du) \nabla_i \nabla_j u$, then for each integer $k \geq 0$ we can bound $|Du|_{k+1, \alpha}$ in terms of $|a^{ij}|_{k, \alpha}$, $|Du|_0$, the bounded geometry of M, N and (when $k \geq 1$) $|Du|_{k-1, \alpha}$.*

Proof. First let's consider the case $k = 0$, which at first glance is literally the Schauder estimate - we just have to make sure the fact that u is a map between manifolds doesn't mess us up. As described in section 2.4.2, it suffices to prove the corresponding result for the components of u in geodesic normal coordinates. Fixing an $R_0 < \min(\iota_M, |Du|_0 \iota_N)$ so that SVD normal coordinates of radius R_0 are injective at any base point, the flow becomes

$$\partial_t u^\alpha = a^{ij} \left(\partial_i \partial_j u^\alpha + \Gamma_{\beta\gamma}^\alpha \partial_i u^\beta \partial_j u^\gamma - \Gamma_{ij}^k \partial_k u^\alpha \right).$$

Since $g_{ij} = \delta_{ij} + O(|x|^2)$, $g^{\alpha\beta} = \delta^{\alpha\beta} + O(|x|^2)$ with constants depending only on curvature bounds and $|Du|_0$, the Christoffel symbols have C^α norm bounded by a constant depending only on R_0 , curvature bounds and $|u|_1$; so applying the Schauder estimate 2.4.19 we get a $C^{2, \alpha}$ estimate for the components of u on $Q(R_0/2)$ with the desired dependencies. By compactness the curvature bound can be made uniform, so this implies a $C^{1, \alpha}$ estimate for Du on each normal coordinate system with the same constant, and thus we have a global estimate.

The first step towards generalizing this to higher derivatives is computing the evolution equation for $\nabla^k u$. We can do this by differentiating the defining equation $Pu = 0$ and commuting the covariant derivatives past the parabolic operator. Thus we first calculate a formula for the commutator $[P, \nabla]$: for any tensor s we have an equation of the form⁴

$$[P, \nabla] s = [\nabla, a^{ij} \nabla_i \nabla_j] s = \nabla a * \nabla^2 s + a * R * \nabla s.$$

Using this repeatedly to calculate $[P, \nabla^k] u$ and expanding all derivatives using the product

⁴For tensors T, S , we use the notation $T * S$ to refer some quantity constructed by taking contractions of $T \otimes S$, with an important consequence being that $|T * S| \leq |T| |S|$.

rule, we find an evolution equation of the form

$$P\nabla^k u = \sum_{\substack{a \geq 1, b \geq 2 \\ a+b \leq k+2}} C_{ab} \nabla^a a * \nabla^b u + \sum_{\substack{a \geq 0, b \geq 1, c \geq 0 \\ a+b+c=k}} C_{abc} \nabla^a a * \nabla^b u * \nabla^c R$$

for some constants C_{ab}, C_{abc} depending only on k . Note that we always have $0 \leq a \leq k$ and $1 \leq b \leq k+1$; so all the terms $\nabla^a a$ appearing have C^α norm bounded by $|a^{ij}|_{k,\alpha}$. The terms with derivatives of u of order k and below thus have C^α norm bounded by the given dependencies, and the terms with derivatives of order $k+1$ can be expressed as combinations of derivatives of $\nabla^k u$ by C^α -controlled coefficients. Thus $s = \nabla^k u$ satisfies an equation of the form

$$\partial_t s^I - a^{ij} \nabla_i \nabla_j s^I - b_{Ji}^I \nabla_i s^J = f^I$$

where a^{ij}, b_{Ji}^I, f^I have controlled C^α norm; so applying the Schauder estimate as in the $k=0$ case yields the desired bound. \square

With this lemma established, the bootstrap argument is easy:

Proposition 5.4.2. *Assume all derivatives of the coefficients $a : J^1(M, N) \rightarrow \text{Sym}^2 TM^*$ are bounded. Then each of the $C^{k,\alpha}$ norms $|Du|_{k,\alpha}$ can be estimated solely in terms of $k, |a^{ij}|_{0,\alpha}, |Du|_0$ and the geometry of M, N .*

Proof. The case $k=0$ is immediate from Lemma 5.4.1. Once we have controlled $|Du|_{j,\alpha}$ for all $j \leq k$, our assumptions on a imply $C^{k,\alpha}$ control of $a^{ij}(Du)$; so applying the lemma gives control of $|Du|_{k+1,\alpha}$. Inducting on k completes the proof. \square

As a corollary of this result along with Theorem 4.2.3 and Theorem 5.3.6, we obtain C^k estimates for the flow in Theorem 1.0.1.

Chapter 6

Existence, Uniqueness and Long-Time Behaviour

We've come all this way and all we've seen are estimates. This is the way of the (PDE) world: given the right a priori estimates, anyone in the know can establish existence, uniqueness, and all that. Despite this, we will dedicate an entire chapter to what a dismissive editor might title *Corollaries of Estimates*. Parts of this chapter follow immediately from standard theory and are simply included for completeness of the argument, while others need a small amount of original work but are not particularly difficult.

The existence results are very much standard - any reader with significant experience in nonlinear evolution equations should find nothing of note here. Short-time existence is a general property of parabolic systems requiring only very mild assumptions, and improving this to long-time existence is very easy given the regularity estimates we established earlier. We will see an explicit proof of long-time existence using the Schauder estimates and the Arzela-Ascoli theorem, which is a standard technique for quasilinear PDE but is rarely spelled out in full.

Results on the long-time behaviour of evolution equations tend to be more specific to the situation at hand, but the methods we use to attain them should be familiar: we can get essentially everything we need to prove Theorem 1.0.1 using energy (i.e. Sobolev norm) estimates.

Compared to the matters of existence, regularity and convergence, the uniqueness and well-posedness of the initial-value problem is explicitly discussed with less frequency in geometric analysis, where it is only sometimes a concern: we are often only interested in the solution to the steady-state problem produced by long-time convergence. It also seems to be well-known that in favourable settings (e.g smooth flows of maps between compact manifolds), well-posedness always follows once we have long-time existence and convergence. This is not too hard to justify: if we dig in to the proofs of short-time existence that we so often take for granted, it turns out that the kinds of boundary conditions we impose in order to make the existence theory work are usually exactly what we need for

uniqueness, and of course short-time uniqueness implies long-time existence¹. Likewise, achieving continuous dependence on initial data is often just a matter of reinterpreting estimates we already have - we will see that the C^k estimates obtained in the previous chapter provide us with well-posedness in the smooth topology. Indeed, the proof of Proposition 6.4.3 will use nothing special about the structure of the flow - all we need are some loose regularity assumptions on the coefficients and C^k estimates.

For some of the calculations in this section, we will be simultaneously considering multiple solutions of a flow, so the notation $P = \partial_t - a^{ij} \nabla_i \nabla_j$ will become ambiguous. We thus introduce the notation $P_u = \partial_t - a^{ij}(Du) \nabla_i \nabla_j$ for the linear operator obtained by freezing out the nonlinearity of P at the map u , and occasionally a_u^{ij} or a_u as shorthand for $a^{ij}(Du)$.

6.1 Short-time Existence and Uniqueness

The local existence of solutions to nonlinear parabolic systems has long been somewhat of a folk theorem. It holds in a wide array of general settings under very mild assumptions, but the proof is quite technical; so it is often difficult to find a source proving it in a general enough form to solve one's problem. In recent years this has been somewhat remedied by various authors - see e.g. the papers of Sharples [Sha04] or Mantegazza and Martinazzi [MM11]; or the thesis chapter of Baker [Bak11, Chapter 3].

Despite these works, however, for the purposes of establishing short-time existence for a geometric map flow we still do not have an existence theorem that we can just cite and forget, since we are dealing with maps $M \rightarrow N$ rather than maps into \mathbb{R}^n or sections of a vector bundle. One way to deal with this would be to develop the existence theory for maps by choosing an atlas on N and patching together local coordinate solutions, but this is quite involved. Instead, following the approach of [LW08] for harmonic map heat flow, we have the following plan: embed our target N in some Euclidean space \mathbb{R}^ℓ , evolve our map by some flow defined for maps into \mathbb{R}^ℓ , and check that the resulting solution stays inside N and solves our original flow equation. For this section, we suppose that M, N are smooth Riemannian manifolds (of any dimension) with M compact, and that $a : J^1(M, N) \rightarrow \text{Sym}^2 TM$ is a smooth map of fibre bundles over M .

Remark. This last assumption does not strictly encompass the flows we are interested in, since their coefficients $a^{ij}(Du)$ have singularities as $Du \rightarrow 0$. Since we are only interested in short-time existence starting from immersions, this is not really a problem: the initial data must have singular values bounded from below by some ϵ ; so we can simply modify the coefficients to be smooth while leaving them unchanged for singular values greater than $\epsilon/2$. By compactness we know that the resulting solution to the modified flow will have $\sigma(Du) > \epsilon/2$ for some short time, and thus will solve the original flow.

¹If there are two distinct solutions with the same initial data, there must be a maximal interval $[0, T]$ on which they agree; so short-time uniqueness is violated on $[T, T + \epsilon)$.

We invoke the Nash embedding theorem, which gives us an isometric embedding $\iota : (N, g_N) \rightarrow \mathbb{R}^\ell$, and study the evolution of the map $U = \iota \circ u : M \rightarrow \mathbb{R}^\ell$. Recall that in this embedded setting, we have a splitting

$$\iota^* T\mathbb{R}^\ell = TN \oplus TN^\perp$$

with normal projection $\pi : \iota^* T\mathbb{R}^\ell \rightarrow TN$. Along ι , the Riemannian connections

$$D : \Gamma(TN) \rightarrow \Gamma(TN^* \otimes TN)$$

of N and

$$\bar{D}|_\iota : \Gamma(TN) \rightarrow \Gamma(TN^* \otimes \iota^* T\mathbb{R}^\ell)$$

of \mathbb{R}^ℓ are related by $\bar{D}_X Y = D_X Y + A(X, Y)$, where

$$A = \bar{D}^\perp \in \Gamma(TN^* \otimes TN^* \otimes TN^\perp)$$

is the second fundamental form of ι . In order to translate our flow equation for u into one for U , the first thing we need is a formula for the harmonic map Laplacian Δu in terms of ΔU .

Proposition 6.1.1. *The covariant second derivative of U splits into TN and TN^\perp components via the formula*

$$\nabla_i \nabla_j U = D\iota(\nabla_i \nabla_j u) + A(\nabla_i u, \nabla_j u).$$

Proof. Fix the standard Euclidean coordinates on \mathbb{R}^ℓ , so that in particular we have $\nabla_i \nabla_j U = \nabla_i \nabla_j (U^a) \partial_a$. Now compute in arbitrary coordinates x^i on M and y^α on N :

$$\begin{aligned} \nabla_i \nabla_j (U^a) &= \partial_i \partial_j (\iota^a \circ u) - \Gamma_{ij}^k \partial_k (\iota^a \circ u) \\ &= \partial_i \left(\frac{\partial \iota^a}{\partial u^\alpha} \frac{\partial u^\alpha}{\partial x^j} \right) - \Gamma_{ij}^k \frac{\partial \iota^a}{\partial u^\alpha} \frac{\partial u^\alpha}{\partial x^j} \\ &= \frac{\partial^2 \iota^a}{\partial u^\alpha \partial u^\beta} u_i^\beta u_j^\alpha + \frac{\partial \iota^a}{\partial u^\gamma} \nabla_i \nabla_j (u^\gamma) \\ &= \frac{\partial^2 \iota^a}{\partial u^\alpha \partial u^\beta} u_i^\beta u_j^\alpha - \frac{\partial \iota^a}{\partial u^\gamma} \Gamma_{\alpha\beta}^\gamma u_i^\alpha u_j^\beta + \frac{\partial \iota^a}{\partial u^\gamma} (\nabla_i \nabla_j u)^\gamma. \end{aligned} \quad (6.1.1)$$

Now note that

$$\begin{aligned} A(\partial_\alpha, \partial_\beta) &= \bar{D}_{D\iota(\partial_\alpha)} D\iota(\partial_\beta) - D\iota(D_\alpha \partial_\beta) \\ &= \frac{\partial^2 \iota}{\partial y^\alpha \partial y^\beta} - \Gamma_{\alpha\beta}^\gamma \frac{\partial \iota}{\partial y^\gamma}, \end{aligned}$$

so composing with u and contracting with $u_i^\alpha u_j^\beta$ we recognize the first two terms of (6.1.1) as $A_{\alpha\beta} u_i^\alpha u_j^\beta = A(\nabla_i u, \nabla_j u)$. \square

Allowing u and thus U to depend on time, we can now express our flow in terms of U : we have $\partial_t u = a^{ij}(Du) \nabla_i \nabla_j u$ if and only if

$$\partial_t U = a^{ij}(DU) (\nabla_i \nabla_j U - A(\nabla_i U, \nabla_j U)). \quad (6.1.2)$$

Note that our notation is obscuring a technical issue here - a^{ij} and A are defined only on the domains $J^1(M, N)$ and $TN \otimes TN$ respectively. This still makes sense as an equation for solutions $U = \iota \circ u$ if we regard these domains as subsets of $J^1(M, \mathbb{R}^\ell)$ and $T\mathbb{R}^\ell \otimes T\mathbb{R}^\ell$; but if we want to treat this as a PDE for the unknown U , then we need to extend a^{ij} and A to the enlarged domains.

For the second fundamental form, there is (at least in a neighbourhood of N) a natural way to do this. Let N_δ be an open neighbourhood of N in \mathbb{R}^ℓ small enough that the nearest-point projection map $\Pi : N_\delta \rightarrow N$ is smooth, and note that when restricted to TN , $D\Pi$ is just the orthogonal projection $\pi \in \Gamma((\iota^* T\mathbb{R}^\ell)^* \otimes TN)$.

Lemma 6.1.2. *The second fundamental form $A \in \Gamma(TN^* \otimes TN^* \otimes \iota^* T\mathbb{R}^\ell)$ and the restricted derivative $\bar{D}\pi|_{TN} = \bar{D}(D\Pi|_{TN})|_{TN} \in \Gamma(TN^* \otimes TN^* \otimes \iota^* T\mathbb{R}^\ell)$ are equal.*

Proof. For arbitrary $X \in \Gamma(TN)$ we have $\pi(X) - X = 0$. Differentiating this in the $Y \in \Gamma(TN)$ direction, we find $(\bar{D}_Y \pi)(X) + \pi(\bar{D}_Y X) = \bar{D}_Y X$; i.e. $(\bar{D}_Y \pi)(X) = \bar{D}_Y X - D_Y X = A(X, Y)$ as desired. \square

Thus we will simply replace $A(\nabla_i U, \nabla_j U)$ with any extension of the smooth section $\bar{D}D\Pi \in \Gamma(T_2^1 N_\delta)$ to $\tilde{A} \in \Gamma(T_2^1 T\mathbb{R}^\ell)$. This particular extension is a very sensible one to choose, as we will see shortly.

For the coefficients a^{ij} , note that $J^1(M, N)$ is a smooth submanifold of $J^1(M, \mathbb{R}^\ell)$ and simply choose any smooth extension to the latter. In the case where we're imposing isometry invariance there is a canonical choice - just let $\tilde{a}^{ij}(DU) = \sum_a F_a(\sigma) e_a^i e_a^j$ where the σ are the singular values of $DU : T_p M \rightarrow \mathbb{R}^\ell$. Thus we now have an honest system of scalar PDE

$$\partial_t U^a = \tilde{a}^{ij}(DU) \nabla_i \nabla_j U^a - \tilde{a}^{ij}(DU) \tilde{A}_{bc}^a(U) \nabla_i U^b \nabla_j U^c \quad (6.1.3)$$

for the unknowns $U^1, \dots, U^\ell : M \rightarrow \mathbb{R}$.

N.B. In this chapter, we use \tilde{a}^{ij} to denote the extension of a^{ij} , which is *not* in the sense it was used in section 3.4.3.

Before we apply the existence theory, we need to confirm that if our initial data has image in N then this stays true for the corresponding solution. Once we know this, the existence of a solution U to (6.1.3) will give us the existence of a solution u to our flow. This is where our choice of \tilde{A} will be important.

Lemma 6.1.3. *If $U : M \times [0, T) \rightarrow \mathbb{R}^\ell$ is a solution to (6.1.3) with initial data $U|_{M \times \{0\}} = \iota \circ u_0$ for some $u_0 : M \rightarrow N$, then $U(M \times [0, T)) \subset \iota(N)$.*

Proof. If we let $V = \Pi \circ U - U$, then the deviation function $\rho = \frac{1}{2} |V|^2$ is smooth, positive and is identically zero at the initial time, and we just need to show that it remains zero. By

continuity and compactness we know that if ρ ever becomes nonzero, it must be nonzero at some time t when $u(M \times \{t\}) \subset N_\delta$; so we can assume u stays inside N_δ so that Π is smooth and $\tilde{A} = \overline{D}D\Pi$. First compute

$$\begin{aligned}\partial_t V &= (D\Pi - I)(\partial_t U) \\ \nabla_i \nabla_j V &= (D\Pi - I)(\nabla_i \nabla_j U) + \overline{D}D\Pi(\nabla_i U, \nabla_j U)\end{aligned}$$

and thus (using $\tilde{A} = \overline{D}D\Pi$ and (6.1.3)) we have

$$(\partial_t - \tilde{a}^{ij} \nabla_i \nabla_j) V = -D\Pi \left(\tilde{a}^{ij} \tilde{A}(\nabla_i U, \nabla_j U) \right).$$

The deviation thus evolves by

$$(\partial_t - \tilde{a}^{ij} \nabla_i \nabla_j) \rho = -\langle V, D\Pi(\tilde{a}^{ij} A(\nabla_i U, \nabla_j U)) \rangle - \tilde{a}^{ij} \langle \nabla_i V, \nabla_j V \rangle.$$

The first term vanishes because V is normal to N and $D\Pi$ projects on to TN , so we are left with

$$(\partial_t - \tilde{a}^{ij} \nabla_i \nabla_j) \rho = -\tilde{a}^{ij} \nabla_i V^a \nabla_j V^a \leq 0.$$

But ρ is non-negative by definition, so the maximum principle yields $\rho = 0$; i.e. U stays in N . \square

Now we state the general local existence theorem for parabolic systems from [Bak11, Main Theorem 5], specialized to the case of second-order equations:

Theorem 6.1.4. *Let E be a vector bundle over M . Consider the initial value problem*

$$\begin{cases} P(U) := \partial_t U - F(x, t, U, \nabla U, \nabla^2 U) = 0 & \text{in } E \times (0, \epsilon), \\ U(M, 0) = U_0 \end{cases}$$

with U_0 a smooth section of E . Define the linearized operator

$$(\partial P[U_0] V)^a = \partial_t V^a - \sum_{|I| \leq 2} A_b^{aI}(x, t) \nabla_I V^b$$

where $A_b^{aI}(x, t) = \frac{\partial F^a}{\partial (\nabla_I U^b)}(x, t, U_0(x), \nabla U_0(x), \nabla^2 U_0(x))$. If F is smooth and the linearized coefficients satisfy the conditions

1. the leading coefficient satisfies the symmetry condition $A_b^{aij} = A_a^{bji}$
2. the leading coefficient satisfies a Legendre-Hadamard condition $A_b^{aI} \xi_I \eta_a \eta^b \geq \lambda |\xi|^2 |\eta|^2$
3. there is a constant $C < \infty$ such that $\sum_{|I| \leq 2} \|A^I\|_{C^\alpha(E)} < C$

then there is a time $\epsilon > 0$ and a unique smooth solution $U \in \Gamma(E \times [0, \epsilon))$ to the initial value problem.

We now have everything we need to apply this to flows of maps.

Proposition 6.1.5. *Suppose $a : J^1(M, N) \rightarrow \text{Sym}^2 TM$ is a smooth map of bundles over M with finite $C^{1,\alpha}$ norm and satisfying a uniform ellipticity condition $a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2$. Then for any smooth map $u_0 : M \rightarrow N$, there is a time $\epsilon > 0$ such that the initial value problem*

$$\begin{cases} \partial_t u = a^{ij}(Du)\nabla_i\nabla_j u & \text{on } M \times [0, \epsilon) \\ u(M, 0) = u_0 \end{cases}$$

has a unique solution.

Proof. As discussed earlier in this section, fix an embedding $\iota : N \rightarrow \mathbb{R}^\ell$ and extend a, A to \tilde{a}, \tilde{A} . Then we would like to apply 6.1.4 with $E = M \times \mathbb{R}^\ell$,

$$F^a = \tilde{a}^{ij}(DU) \left(\nabla_i \nabla_j U^a - \tilde{A}_{bc}^a(U) \nabla_i U^b \nabla_j U^c \right)$$

and $U_0 = \iota \circ u_0$ - we just need to check the conditions. The linearized operator is

$$\begin{aligned} (\partial P[U_0] V)^a &= \partial_t V^a - \tilde{a}^{ij}(DU_0) \nabla_i \nabla_j V^a - \frac{\partial \tilde{a}^{ij}}{\partial u_k^\beta}(DU_0) \nabla_i \nabla_j U_0^a \nabla_k V^\beta \\ &\quad + 2\tilde{A}_{bc}^a(U_0) \nabla_i U_0^b \nabla_j V^c + \frac{\partial \tilde{A}_{bc}^a}{\partial U^l}(U_0) \nabla_i U_0^b \nabla_j U_0^c V^l; \end{aligned}$$

in particular the leading coefficient is $A_b^{aij} = \delta_b^a a^{ij}(DU_0)$, so the symmetry condition is satisfied and the Legendre-Hadamard condition reduces exactly to the ellipticity assumption. Finally, the assumed Hölder bound on the coefficients is exactly what we require to satisfy condition 3; so the theorem provides us with a short-time solution to $\partial_t U = a^{ij}(DU_0) \nabla_i \nabla_j U$. Lemma 6.1.3 then tells us that U stays in N , so defining $u = \iota^{-1} \circ U$ (which is smooth because ι is an embedding) we have our desired solution. Since any other solution v to the map flow would yield a solution $\iota \circ v$ to our embedded flow, Theorem 6.1.4 also provides us with uniqueness. \square

6.2 Long-time Existence

We will now show the higher derivative estimate (Proposition 5.4.2) allows us to improve short-time existence to long-time existence:

Theorem 6.2.1 (Global Existence Theorem). *Let M, N be flat surfaces. For any $u_0 \in \text{Diff}(M, N)$, there exists a solution $u : [0, \infty) \rightarrow \text{Diff}(M, N)$ of the flow $\partial_t u = (\sigma_1 + \sigma_2)^{-2} \Delta u$ (1.0.1) with initial data $u(\cdot, 0) = u_0$.*

Proof. Let $T \in [0, \infty]$ be the maximal time of existence; i.e. the supremum of times $T' \in [0, \infty)$ such that there exists a solution of the flow on $[0, T']$ with initial data u_0 .

We want to show $T = \infty$; so assume on the contrary that $T < \infty$, so we have a solution on $[0, T)$ that cannot be extended further. Let $t_i \rightarrow T$ be an arbitrary sequence of times in $[0, T)$ and consider the sequence of diffeomorphisms $u_i = u(\cdot, t_i)$. By our estimates (Theorem 4.2.3, Theorem 5.3.6, Proposition 5.4.2) we know that the u_i have uniformly controlled C^k norm for each $k \in \mathbb{N}$. Since uniform C^1 bounds imply equicontinuity, the Arzela-Ascoli theorem gives us a subsequence on which u_i converges uniformly; call the limit u_T . The uniform C^2 bounds imply equicontinuity of the derivatives, so passing to another subsequence we can make Du_i converge uniformly. Proceeding in this fashion for all k and then taking the diagonal subsequence, we find a subsequence of u_k that converges in C^∞ to u_T . The continuity of $t \mapsto u(\cdot, t)$ in the C^k topology implies that any two such sequences will be co-Cauchy in each C^k ; so there is a universal $u_T \in C^\infty(M, N)$ such that every sequence of times $t_k \rightarrow T$ has a subsequence on which $u(\cdot, t_k) \rightarrow u_T$ smoothly. Since our diffeomorphism-preserving estimate tells us that Du stays in the closed subset of $J_{\text{inv}}^1(M, N)$ defined by $\{\sigma_{\min}(\cdot) \geq \inf \sigma(Du_0)\}$, the C^1 convergence $u_k \rightarrow u_T$ implies that u_T is a diffeomorphism. Applying the short-time existence theorem Proposition 6.1.5 with initial data u_T gives us a solution on $[T, T + \epsilon)$ which we can concatenate with u to get a smooth solution $\bar{u} : [0, T + \epsilon) \rightarrow \text{Diff}(M, N)$, contradicting the fact that T is maximal. \square

6.3 Limiting Behaviour

As described in the introduction, a strong motivation for studying geometric heat flows is to take the long-time limit to produce solutions of the corresponding steady-state equation, along with a deformation retract of the space of initial data on to the space of steady-state solutions. In the case of (1.0.1), the steady-state equation $0 = \partial_t u = (\sigma_1 + \sigma_2)^{-2} \Delta u$ reduces (for diffeomorphisms) to the harmonic map equation $\Delta u = 0$; so if we can establish convergence as $t \rightarrow \infty$ then the flow produces a smooth path from an arbitrary $u_0 \in \text{Diff}(M, N)$ to a harmonic diffeomorphism. As in the case of the harmonic map heat flow, we can establish this convergence using integral estimates. The following argument appears mostly as in [AC16]. We will consider only the case of isotropic flows $\partial_t u = F \Delta u$, since they are (loosely) L^2 -gradient-like for the *Dirichlet energy* $E(u) = \frac{1}{2} \int |Du|^2$.

Proposition 6.3.1. *Suppose that $u : [0, \infty) \rightarrow \text{Diff}(M, N)$ is a solution of $\partial_t u = F \Delta u$ such that*

- *there are positive constants $\lambda, \Lambda, \lambda_-, \lambda_+$ such that $\lambda \leq F \leq \Lambda$ and $\lambda_- \leq \sigma(Du) \leq \lambda_+$, e.g. as obtained from Theorem 4.2.3; and*
- *there are constants C_k such that $|u(\cdot, t)|_k \leq C_k$ for each $k \in \mathbb{N}$, e.g. as obtained from Proposition 5.4.2.*

Then there exists a sequence $t_k \rightarrow \infty$ such that $u_k := u(\cdot, t_k)$ converges smoothly to a harmonic diffeomorphism u_∞ .

Proof. As mentioned above, the energy $E(t) = E(u(\cdot, t))$ is monotone under such a flow: applying the product rule, evolution equation and integration by parts yields

$$E'(t) = \int \langle Du, \nabla_t Du \rangle = \int \langle Du, D\partial_t u \rangle = - \int F |\Delta u|^2.$$

Thus $E(t)$ is a non-negative non-increasing function, so it must converge to some non-negative value $E(\infty) \leq E(0)$. This implies $\int_0^\infty -E'(t) dt < \infty$, so there must be some sequence of times $t_k \rightarrow \infty$ such that $E'(t_k) \rightarrow 0$. Since $0 \leq \int |\Delta u|^2 \leq \frac{1}{\lambda} \int F |\Delta u|^2 = -\frac{1}{\lambda} E'(t)$, we see that $\|\Delta u(\cdot, t_k)\|_2 \rightarrow 0$. As in the proof of Theorem 6.2.1, the C^k bounds and the Arzela-Ascoli theorem allow us to pass to a subsequence on which $u(\cdot, t_k)$ converges smoothly to some limit u_∞ , and the lower bound $\sigma(Du) \geq \lambda_-$ enforces $u_\infty \in \text{Diff}(M, N)$; so in particular $\|\Delta u_\infty\|_2 = \lim_k \|\Delta u_k\|_2 = 0$ and thus u_∞ is harmonic. \square

This convergence on a subsequence requires quite weak hypotheses to work - notice that the nonlinearity of F and the geometry of N are not important. Improving this to convergence for all time, however, is somewhat difficult - the coefficient F along with extra terms coming from the curvature of N mean that the energy is not convex as a function of time, unlike the case of the scalar heat equation. Thus we need to find some other way to get a decay estimate for $\partial_t u$. The flow equation (and our bound on F) suggests that we can do this by estimating Δu ; so we will investigate the evolution of the quantity $q(t) = \frac{1}{2} \int |\Delta u(\cdot, t)|^2$. Similar calculations to those we did for E yield

$$q'(t) = \int \langle \Delta u, \nabla_t \Delta u \rangle = \int \langle \Delta u, [\nabla_t, \Delta] u \rangle - \int F |\nabla \Delta u|^2 - \int \langle \nabla \Delta u, \nabla F \otimes \Delta u \rangle. \quad (6.3.1)$$

The commutator term arises because (unlike for the first-order energy) higher-order derivatives do not commute, so we will pick up some extra curvature term. Using Proposition 3.1.2 and Proposition 3.1.3 we can compute it as

$$\begin{aligned} [\nabla_t, \Delta] u^\alpha &= g^{ij} (\nabla_t \nabla_i \nabla_j u^\alpha - \nabla_i \nabla_t \nabla_j u^\alpha) \\ &= g^{ij} \nabla_t u^\gamma \nabla_i u^\delta \nabla_j u^\beta R_{\gamma\delta}{}^\alpha{}_\beta; \end{aligned}$$

so substituting $\nabla_t u^\gamma = F \Delta u^\gamma$ and writing this in the SVD frames we get

$$\int \langle \Delta u, [\nabla_t, \Delta] u \rangle = \int F \sum_{i,\alpha} (\sigma_i \langle \Delta u, v_\alpha \rangle)^2 K_N(v_i \wedge v_\alpha);$$

so the non-positive sectional curvature of N (a standard assumption in harmonic map heat flow) would let us neglect this term, while otherwise it contributes growth bounded by a multiple of $q(t)^2$. However, the ∇F term is quite problematic - this will end up contributing $\nabla^2 u$ terms, and the best we can do to control this in terms of q is $\|\nabla^2 u\|_2 \lesssim \|\Delta u\|_2 + \|R * Du\|_2$. While there may be some subtle estimate that works under certain curvature assumptions, we will simply give the version we need for the proof of Theorem 1.0.1.

Lemma 6.3.2. *If M, N are compact flat surfaces and $u : [0, T) \rightarrow \text{Diff}(M, N)$ satisfies the hypotheses of Proposition 6.3.1, then $q(t) \rightarrow 0$ exponentially as $t \rightarrow \infty$.*

Proof. Since we can freely exchange the order of covariant derivatives, integration by parts shows that $\|\nabla^2 u\|_2 = \|\Delta u\|_2$; so we can derive a nicer analog of (6.3.1) by differentiating $\|\nabla^2 u\|_2^2$ to get

$$q'(t) = - \int F |\nabla^3 u|^2 - \int \dot{F} * \nabla^2 u * \nabla^2 u * \nabla^3 u$$

where \dot{F} denotes the derivative of F with respect to Du . Estimating the latter term with Cauchy-Schwarz and Peter-Paul we get

$$q'(t) \leq -\lambda \|\nabla^3 u\|_2^2 + C \left(\epsilon \|\nabla^2 u\|_4^4 + \frac{1}{\epsilon} \|\nabla^3 u\|_2^2 \right).$$

Applying the Ladyzhenskaya inequality $\|f\|_4^2 \lesssim \|f\|_2 \|\nabla f\|_2$ (which is an instance of Proposition 2.4.12 in dimension 2) to $\|\nabla^2 u\|_4^4$ and choosing an appropriate ϵ we get

$$q'(t) \leq -C_1 (1 - C_2 q(t)) \|\nabla^3 u\|_2^2.$$

Since we know $q(t_k) \rightarrow 0$, there must be some $t^* = t_k$ such that $C_2 q(t^*) < 1/2$ and thus $q'(t^*) \leq 0$; so this inequality is preserved from then on and thus for $t \geq t^*$ we have

$$q'(t) \leq -\frac{C_1}{2} \|\nabla^3 u\|_2^2 \lesssim -q(t)$$

by the Poincaré inequality; so comparison to the corresponding ODE gives $|q(t)| = q(t) \leq q(t^*) e^{-c(t-t^*)}$. \square

Thus we know $\partial_t u$ converges exponentially to zero in L^2 . This along with C^k estimates is enough to conclude smooth convergence of the flow:

Proposition 6.3.3. *If M, N are compact flat surfaces and $u : [0, T) \rightarrow \text{Diff}(M, N)$ satisfies the hypotheses of Proposition 6.3.1, then $u(t, \cdot) \rightarrow u_\infty$ in C^∞ as $t \rightarrow \infty$.*

Proof. Applying the Gagliardo-Nirenberg inequality (Proposition 2.4.12) $\|f\|_\infty^2 \leq \|Df\|_\infty \|f\|_2$ to $\partial_t u = F \Delta u$, the uniform C^3 bound along with Lemma 6.3.2 imply a uniform exponential decay estimate $|\partial_t u|_0 \leq C e^{-ct/2}$. Thus for any times $b > a \geq 0$ we have

$$|u(\cdot, b) - u(\cdot, a)|_0 \leq \int_a^b C e^{-ct/2} dt = \frac{2C}{c} |e^{-cb/2} - e^{-ca/2}|$$

which converges exponentially to zero as $a, b \rightarrow \infty$. Since we have already established the subconvergence $|u(\cdot, t_k) - u_\infty|_0 \rightarrow 0$, we conclude that in fact $u(\cdot, t) \rightarrow u_\infty$ uniformly as $t \rightarrow \infty$. Fixing a $j \in \mathbb{N}$ and interpolating again we find

$$\|\nabla^j u - \nabla^j u_\infty\|_\infty^2 \lesssim \|u - u_\infty\|_\infty \|\nabla^{2j} u - \nabla^{2j} u_\infty\|_\infty,$$

so the uniform C^{2j} bound allows us to improve the convergence of u to from uniform to C^j . \square

To handle more general settings (higher dimension, arbitrary curvature, etc.), these methods would require a lot of modification. Even just moving to 3 dimensions (while keeping the flatness assumption) prevents Lemma 6.3.2 from going through; and even non-positive curvature makes it difficult to handle the ∇F term. An approach that might be more fruitful in general is to use the linearized stability principle: under mild assumptions, a stationary point of a nonlinear evolution equation on a Banach manifold is asymptotically stable (i.e. is surrounded by an attractive basin) if and only if the linearized equation at the stationary point is stable. This would provide convergence as soon as we had subconvergence, e.g. as provided by Proposition 6.3.1, which has very relaxed assumptions compared to Proposition 6.3.3. For example, Theorem 9.1.2 of [Lun95] works directly for a torus of any dimension once we restrict to a class of diffeomorphisms with the same centre of mass as measured on some fundamental domain.

6.4 Well-posedness of the Initial Value Problem

The final property of initial-value problems that we would like to address is whether or not they are well-posed; that is, in addition to existing and being unique, the solution should be continuously dependent upon the initial data in some sense. These requirements are very natural from the applied/physical point of view: if you were planning on using a system of PDE to model some phenomenon then you need both uniqueness and continuous dependence in order to make useful predictions - uniqueness so that only a single prediction is made, and continuous dependence so that the effects of uncertainty in measurements of initial data on the uncertainty of the predictions can be estimated. In geometric and topological applications of flows, well-posedness is important because it strengthens the implications of our existence and convergence theorem - once we have well-posedness, the flow gives us a deformation retract of the space of initial data on to the space of long-time limits, instead of just individual path connections.

In the context of smooth solutions to heat-type equations on compact domains, well-posedness in C^∞ is fairly generic: the kind of estimates that are developed to establish existence and regularity results are typically all we need to write down a general proof of well-posedness. For simplicity, we will also assume compactness of the target; but something like bounded geometry of all orders (i.e. positive injectivity radius plus uniform bounds on all derivatives of the curvature tensor) would suffice in this setting.

To illustrate this genericity, we will formulate the proof for a wide class of flows of immersions. However, since our nonlinearities are typically singular as $Du \rightarrow 0$, we will need to build in the fact that the flows we are interested in avoid these singularities. Thus in addition to assuming that N is a compact Riemannian manifold isometrically embedded in some \mathbb{R}^ℓ , that the coefficients $a : J_{\text{inv}}^1(M, N) \rightarrow \text{Sym}^2 TM$ are smooth and positive-

definite, and the existence of global solutions to the flow, we require an abstract version of our diffeomorphism-preserving estimate:

Assumption 6.4.1. *There exists an exhaustion \mathcal{O} of $J_{\text{inv}}^1(M, N)$ by compact sets $\Omega \in \mathcal{O}$ that are preserved by the flow generated by a ; i.e. if u is a solution and $j^1u(M \times \{0\}) \subset \Omega \in \mathcal{O}$ then $j^1u(M \times [0, T]) \subset \Omega$.*

The examples to have in mind here are the flows we found in Chapter 4 that preserve upper and lower bounds on the singular values, for which we can take $\mathcal{O} = \{\Omega_\Lambda : \Lambda \geq 1\}$ with $\Omega_\Lambda = \{\xi \in J_x^1(M, N)_y : \Lambda^{-1} \leq \sigma_a(\xi) \leq \Lambda\}$.

The compactness of each Ω implies that $a|_\Omega$ can be smoothly extended to all of $J^1(M, N)$, and thus that $a|_\Omega$ is Lipschitz in the sense that there exists a constant $C = C(\Omega) < \infty$ such that $|a(\xi) - a(\zeta)| \leq C|\xi - \zeta|$ for all $\xi, \zeta \in \Omega$. It also gives us uniform ellipticity: if we are interested in initial data coming from a given Ω , we have $a^{ij} \geq \lambda_\Omega g^{ij}$ where

$$\lambda_\Omega = \inf_{\substack{(x,y,\xi) \in \Omega \\ v \in UT_x M}} \langle a(\xi), v \otimes v \rangle$$

is positive by compactness.

We will also need control on higher derivatives, so we assume our flow has higher C^k estimates:

Assumption 6.4.2. *For any solution u and each integer $k \geq 1$, the C^k norm $|u|_k$ is bounded by a constant depending only on k , a^{ij} and $|u(\cdot, 0)|_k$.*

Since we have established existence and uniqueness earlier in this chapter, it now makes sense to talk about the *solution operator* $\Phi : \text{Imm}(M, N) \times [0, T] \rightarrow C^\infty(M, N)$, which takes an initial condition and produces the corresponding solution; i.e. $\Phi(u_0, \cdot)$ is defined as the unique solution u of the flow satisfying $u(\cdot, 0) = u_0$. The question of well-posedness now comes down to showing that Φ is continuous in some useful topology. The nonlinear nature of our equations make unconditional well-posedness in any given C^k difficult to obtain: for example, applying the maximum principle to the difference of two solutions u, v yields

$$|\Phi(u_0, t) - \Phi(v_0, t)|_0 \leq e^{C(|v_0|_2)t} |u_0 - v_0|_0,$$

so we get continuity in the C^0 topology only if we restrict to a domain with bounded C^2 norm (which will not be open in the C^0 topology). This suggests that we work in the C^∞ topology instead: if a sequence of initial data converges in the C^∞ topology, then it will converge and thus be bounded in every C^k , so when showing that the corresponding solutions converge in C^k we will have the required C^{k+2} bound available.

Proposition 6.4.3. *Assume a^{ij} satisfies Assumption 6.4.1 and Assumption 6.4.2 for all $k \in \mathbb{N}$. If $f_m \rightarrow f$ is a smoothly convergent sequence of immersions $M \rightarrow N$, then the corresponding solutions $u_m = \Phi(f_m)$ of the flow $\partial_t u_m^\alpha = a^{ij}(Du_m) \nabla_i \nabla_j u_m^\alpha$ converge smoothly to $u = \Phi(f)$.*

Proof. Since $u_m \rightarrow u$ in C^1 and we have some $\Omega' \in \mathcal{O}$ containing the image of $j^1 u$, by passing to a larger $\Omega \in \mathcal{O}$ (such that Ω' is compactly contained in the interior of Ω) we can discard finitely many terms to make sure the image of every $j^1 u_m$ lies in Ω , so there is some uniform ellipticity constant λ for all the operators P_u, P_{u_m} . Extending $a|_\Omega$ smoothly from this compact set, we can assume that all derivatives of a are Lipschitz. To prove the proposition, it suffices to show that $|\nabla^k u_m - \nabla^k u|_0 \rightarrow 0$ for arbitrary $k \in \mathbb{N}$. Now, take two solutions $v := u_m$ and u , let $w = u - v$ and define the deviation $h = \frac{1}{2} \sum_{l=0}^k |\nabla^l w|^2$, so that $\sup h \rightarrow 0$ will imply $|w|_k \rightarrow 0$. We see that h satisfies the equation

$$P_u h = \sum_{l=0}^k \left(\left\langle P_u \nabla^l w, \nabla^l w \right\rangle - a^{ij} (Du) \left\langle \nabla_i \nabla^l w, \nabla_j \nabla^l w \right\rangle \right).$$

Bounding the quadratic gradient terms using the ellipticity we can estimate this as

$$P_u h \leq \sum_{l=0}^k \left\langle P_u \nabla^l w, \nabla^l w \right\rangle - \lambda \sum_{l=1}^{k+1} |\nabla^l w|^2. \quad (6.4.1)$$

Since we want to make h small, the last term here is good; so the idea is to use some of it to control the $P_u \nabla^l w$ terms. As in the proof of Lemma 5.4.1, we can iterate the commutator identity

$$[P_u, \nabla] s = [\nabla, a^{ij} \nabla_i \nabla_j s] = \nabla a * \nabla^2 s + a * R * \nabla s$$

to find

$$P_u \nabla^l w = \nabla^l P_u w + \sum_{\substack{a \geq 1, b \geq 2 \\ a+b \leq l+2}} C_{ab} \nabla^a a_u * \nabla^b w + \sum_{\substack{a \geq 0, b \geq 1, c \geq 0 \\ a+b+c=l}} C_{abc} \nabla^a a_u * \nabla^b w * \nabla^c R$$

for some constants C_{ab}, C_{abc} depending only on l . Since (from our assumptions) all derivatives of a on Ω and u on M are bounded in terms of the initial data, when we estimate this we can absorb all the a_u and R terms in to the constants, yielding

$$\sum_{l=0}^k \left\langle P_u \nabla^l w, \nabla^l w \right\rangle \leq \sum_{l=0}^k \left| \left\langle \nabla^l P_u w, \nabla^l w \right\rangle \right| + C \sum_{l=0}^k \sum_{b=1}^{l+1} |\nabla^l w| |\nabla^b w|. \quad (6.4.2)$$

For the second term, the key observation to make is that almost all the terms in the double sum can be controlled by a multiple of h - the only exception is the single term $l = k, b = k + 1$ where we get a multiple of $|\nabla^l w| |\nabla^{l+1} w|$. For the first term, note that $P_u w = (a^{ij} (Du) - a^{ij} (Dv)) \nabla_i \nabla_j v$. Taking any number of derivatives of this and absorbing derivatives of a, u, v in to the constant, we get $|\nabla^l P_u w| \leq C \sum_{b=1}^{k+1} |\nabla^b w|$. Collecting terms and estimating the ones with only derivatives of order k and below using

Young's inequality, (6.4.2) becomes

$$\sum_{l=0}^k \left\langle P_u \nabla^l w, \nabla^l w \right\rangle \leq C \sum_{b=1}^k \left(\left| \nabla^b w \right|^2 + \left| \nabla^b w \right| \left| \nabla^{k+1} w \right| \right).$$

Now, applying the Peter-Paul inequality

$$\left| \nabla^b w \right| \left| \nabla^{k+1} w \right| \leq \frac{C}{4\lambda} \left| \nabla^b w \right|^2 + \frac{\lambda}{C} \left| \nabla^{k+1} w \right|^2$$

and substituting back in to (6.4.1), the $\nabla^{k+1} w$ term cancels with the good term and the lower-order derivatives can be controlled by h , we so we arrive at $P_u h \leq Ch$ for some constant C . Comparing to the corresponding ODE proves $\sup h(\cdot, t) \leq e^{Ct} \sup h(\cdot, 0)$. Since $\sup h = \sum_{i=0}^k \left| \nabla^i (u - u_m) \right|_0^2$ is comparable to $|u - u_m|_k$, this gives the desired result: we have $|u - u_m|_k \rightarrow 0$ as $|f - f_m|_k \rightarrow 0$. \square

6.5 Conclusion

We have finally covered everything we need to complete the proof of our main theorem, which we recall here:

Theorem 1.0.1. *For any flat 2-dimensional tori M, N and any $u_0 \in \text{Diff}(M, N)$, there is a smooth solution $u : [0, \infty) \rightarrow \text{Diff}(M, N)$ of the flow (1.0.1), and $u(t)$ converges smoothly to a harmonic diffeomorphism as $t \rightarrow \infty$. Moreover, the solution map sending (u_0, t) to $u(t)$ is continuous in the C^∞ topology, so the flow constitutes a strong deformation retract of $\text{Diff}(M, N)$ on to the finite-dimensional space of harmonic diffeomorphisms.*

Proof. Thanks to all the work we did in Chapters 4 and 5, we have Theorem 6.2.1, which provides a unique solution $u : [0, \infty) \rightarrow \text{Diff}(M, N)$ of the flow (1.0.1). In Proposition 6.3.3 we showed that this solution converges to a harmonic diffeomorphism. Finally, Proposition 6.4.3 tells us that the solution operator is continuous in the smooth topology. \square

As we have discussed in numerous places throughout the text, there are many directions in which this work could be extended. One goal that is tantalizingly close is to obtain a new proof of Smale's theorem $\text{Diff}(S^2) \simeq O(3)$, or similar results for other surfaces. There are a couple of plausible paths in this direction:

1. Find some replacement for Theorem 4.2.3 that gives singular value bounds under more relaxed curvature assumptions. If we could do this, the rest of the argument would follow immediately, since the estimates in Chapter 5 do not require flatness (so long as we replace Proposition 5.3.5 with Proposition C.0.7). Since equation (1.0.1) is conformally invariant, in the case of the sphere the correct approach might be to study some "conformal derivative" arising naturally from Möbius geometry. Alternatively, there might be some global way of dealing with the curvature - swapping out the

maximum principle for L^p estimates doesn't seem immediately fruitful, but perhaps there is a trick hiding somewhere.

2. “Unwrap” the sphere to a disc (as described in the Introduction) and solve an initial-boundary-value problem for diffeomorphisms of the disc. Since the disc is flat, a combination of Theorem 4.2.3 and a simple barrier argument allows us to obtain the necessary C^1 estimates; but unfortunately, sensible choices of boundary condition seem to prevent the decoupling of the evolution of θ that is crucial in the proof of Theorem 5.3.6. See Appendix B for details. One variant of this idea is to instead stereographically project the sphere to \mathbb{R}^2 and try to flow diffeomorphisms that decay to the identity at infinity.

Extending this program to higher dimensions seems less realistic: in their current forms, both of our central estimates depend essentially upon the two-dimensional structure, so it appears that we would need multiple genuinely new ideas. The evolution equation for θ found in Chapter 5 can be written in an invariant form that seems ripe for generalization: see Appendix C. However, while these equations *make sense* in higher dimension, it is very unclear as to whether they are actually *true* there; and even then, the higher dimension of the orthogonal group means some extra ingredient would be necessary to get a Hölder estimate for the rotational component.

Appendix A

Hölder Spaces on Compact Manifolds

This appendix provides the omitted proofs of the equivalence results in §2.4. Recall that (M, g) is a compact Riemannian manifold.

A.1 C^k spaces

Proposition 2.4.3. *If g, h are Riemannian metrics on M , then the C^k norms $|\cdot|_{k,0;(M,g)}$ and $|\cdot|_{k,0;(M,h)}$ that they generate are equivalent.*

Proof. Let ∇ denote the connection of g , D that of h and $\Gamma = D - \nabla \in \Gamma(T_2^1 TM)$ their difference. Then repeatedly applying the product rule and writing ∇ in terms of Γ allows us to get a sequence of formulae for $\nabla^k f$ in terms of $D^k f$ and Γ :

$$\begin{aligned} \nabla_i f &= D_i f \\ \nabla_j \nabla_i f &= D_j D_i f + \Gamma_{ji}^l D_l f \\ \nabla_k \nabla_j \nabla_i f &= D_k D_j D_i f + \Gamma_{ki}^l D_l D_j f + \Gamma_{kj}^l D_l D_i f \\ &\quad + \left(D_k \Gamma_{ij}^l + \Gamma_{mj}^l \Gamma_{ki}^m + \Gamma_{im}^l \Gamma_{kj}^m - \Gamma_{ij}^m \Gamma_{km}^l \right) D_l f + \Gamma_{ij}^l (D_k D_l f + \Gamma_{kl}^m D_m f) \\ &\vdots \end{aligned}$$

While the exact formulae quickly become very painful to write down, all we care about is the general form: we can always write $\nabla^k f - D^k f$ in the form $\sum_{j=0}^{k-1} c_j(\Gamma, \dots, D^{j-1} \Gamma) D^j f$ for some functions $x \mapsto c_j(\Gamma(x), \dots, D^{j-1} \Gamma(x))$; and the compactness of M guarantees these functions are bounded.

Since M is compact we know that all derivatives of Γ are bounded in the g -norm. We also know there is some constant C such that $\frac{1}{C^2} h \leq g \leq C^2 h$ - just consider the smooth positive function $G = g(\cdot, \cdot)$ on the compact space $U_h TM = \{v \in TM : |v|_h = 1\}$ and let $C = \sqrt{\max(\sup G, \sup G^{-1})}$. Thus we have

$$C^{-k} \left| \nabla^k f \right|_h \leq \left| \nabla^k f \right|_g \leq C^k \left| \nabla^k f \right|_h$$

and therefore

$$\begin{aligned} \left| \nabla^k f \right|_g &\leq C^k \left| \nabla^k f \right|_h \leq C^k \left(\left| D^k f \right|_h + \sum_{j < k} c_j(\Gamma, \dots, D^{j-1} \Gamma) \left| D^j f \right|_h \right) \\ &\leq C^k \left(1 + \sum_{j < k} |c_j(\Gamma, \dots, D^{j-1} \Gamma)|_0 \right) |f|_{k,0;(M,h)}; \end{aligned}$$

so $|f|_{k,0;(M,g)} = \sup |\nabla^k f|_g \lesssim |f|_{k,0;(M,h)}$ with constant depending only on g and h . Swapping the role of the metrics and repeating this argument gives us the desired equivalence. \square

A.2 Hölder Spaces

Before we prove Proposition 2.4.7, we will need a few lemmas. Firstly, we quickly note that the injectivity radius ι_g used in the definition of the Hölder norm is not really essential - we can restrict the sample points x, y to be as close as we like and we get an equivalent norm:

Lemma A.2.1. *For all $R \leq \iota_g$ the “restricted Hölder norm” $|f|_{k,\alpha|R} := |f|_k + [\nabla^k f]_{\alpha|R}$ where*

$$[\nabla^k f]_{\alpha|R} = \sup_{\substack{x \neq y \\ d(x,y) < R}} \frac{|\nabla^k f(x) - \tau_{\gamma_{yx}} \nabla^k f(y)|}{d(x,y)^\alpha}$$

is equivalent to $|f|_{k,\alpha} = |f|_{k,\alpha|\iota_g}$.

Proof. One direction is immediate: since we are just modifying the supremum to be over a smaller domain, we cannot possibly increase the norm; i.e. $|f|_{k,\alpha|R} \leq |f|_{k,\alpha}$. Since parallel transport preserves the norm, when $d(x,y) \geq R$ the term in the supremum is at most $2R^{-\alpha} |f|_k$, so we have

$$|f|_{k,\alpha} \leq |f|_k + \max \left(2R^{-\alpha} |f|_k, [\nabla^k f]_{k,\alpha|R} \right) \leq (1 + 2R^{-\alpha}) |f|_{k,\alpha|R}.$$

\square

Adding the Hölder term makes metric independence quite a bit harder to prove in this setting, since we need to deal with the parallel transport. We will first establish the equivalence locally, by considering only the domain of a chart φ . To avoid φ blowing up with respect to the geometry of M , we will add a technical condition that can be achieved by simply shrinking the domain of any chart; i.e. replacing φ with $\varphi|_{\mathcal{U}'}$ for any open $\mathcal{U}' \Subset \mathcal{U}$ (\mathcal{U}' compactly contained in \mathcal{U} , that is $\mathcal{U}' \subset \mathcal{V} \subset \mathcal{U}$ for some compact \mathcal{V}).

Lemma A.2.2. *Let $\varphi : \mathcal{U} \rightarrow \mathbb{R}^m$ be a chart that extends continuously to $\overline{\mathcal{U}} \subset M$ and δ, ∂ be the pullback via φ of the Euclidean metric and connection of \mathbb{R}^m . Then*

$$\left[\nabla^k f \right]_{\alpha; \mathcal{U}, g} \lesssim \left[\partial^k f \right]_{\alpha; \mathcal{U}, \delta} + |f|_k$$

and

$$\left[\partial^k f \right]_{\alpha; \mathcal{U}, \delta} \lesssim \left[\nabla^k f \right]_{\alpha; \mathcal{U}, g} + |f|_k;$$

i. e. $|\cdot|_{k, \alpha; \mathcal{U}, g}$ is equivalent to $|\cdot|_{k, \alpha; \mathcal{U}, \delta}$.

Proof. We will omit the explicit \mathcal{U} from all our norms, and (in view of Proposition 2.4.3) we will not distinguish between the C^k norms of δ and g . From the definition we have

$$\left[\nabla^k f \right]_{\alpha; g} = \sup_{\substack{x \neq y \\ d(x, y) < \iota_g}} \frac{|\nabla^k f(x) - \tau_{yx} \nabla^k f(y)|_g}{d(x, y)^\alpha}$$

where τ_{yx} is parallel translation in the bundle $\otimes^k T^*M$ along the geodesic from y to x . For fixed x, y closer than the injectivity radius, let $\gamma : [0, L] \rightarrow M$ be the unit-speed minimizing geodesic joining y to x and $\xi \in \Gamma(\gamma^*(\otimes^k T^*M))$ the parallel translation of $\nabla^k(y)$ along γ . Using capital Roman indices for the bundle $\otimes^k T^*M$ and lowercase for the coordinate chart φ^i , this means that ξ solves the differential equation

$$\frac{d\xi^A}{ds} + \omega_{Bi}^A(\gamma(s)) \xi^B \frac{d\gamma^i}{ds} = 0$$

where the connection form coefficients ω_{Bi}^A are linear in the Christoffel symbols of g in the coordinates φ^i . Since the extension of φ^{-1} to the closure is a smooth function on a compact set, all derivatives of g are bounded; so taking norms (with respect to δ) and estimating using Gronwall's inequality gives

$$\left| \tau_{yx} \nabla^k f(y) - \nabla^k f(y) \right| = |\xi(1) - \xi(0)| \leq \left| \nabla^k f(y) \right| \left(e^{Cd(x, y)} - 1 \right)$$

where C depends only on $|g|_1$. Now recall that

$$Z := \nabla^k f - \partial^k f = \sum_{|I| < k} c_I(\Gamma, \dots, \partial^{j-1} \Gamma) \partial^I f,$$

so

$$\begin{aligned} \left| \nabla^k f(y) - \nabla^k f(x) \right| - \left| \partial^k f(y) - \partial^k f(x) \right| &\leq |Z(y) - Z(x)| \\ &\leq \sum_{|I| < k} (|c_I|_0 |\partial^I f(x) - \partial^I f(y)| + |c_I(x) - c_I(y)| |f|_{k-1}); \end{aligned}$$

which is bounded by $C_1 |f|_k |x - y|$ for some constant C_1 depending only on $|\partial g|_{k-1}$. Before we proceed, note that the bounds on g imply that we can freely exchange $|\cdot|_\delta$ with $|\cdot|_g$ and

$|\cdot - \cdot|$ with $d(\cdot, \cdot)$ at the expense of a constant. Putting everything we have together, we get

$$\frac{|\tau_{yx} \nabla^k f(y) - \nabla^k f(x)|_g}{d(x, y)^\alpha} \lesssim \frac{|\partial^k f(y) - \partial^k f(x)| + |f|_k (e^{Cd(x, y)} - 1 + C_1 |x - y|)}{|x - y|^\alpha}.$$

Thus restricting to any small radius R we get

$$[\nabla^k f]_{\alpha|R;g} \lesssim [\partial^k f]_{\alpha|R';\delta} + \frac{|f|_k}{|x - y|^\alpha} \left(\frac{e^{CR}}{R} d(x, y) + C_1 |x - y| \right) \lesssim [\partial^k f]_{\alpha|R';\delta} + |f|_k$$

since $(d(x, y) + |x - y|) / |x - y|^\alpha \lesssim R^{1-\alpha}$. Here R' is just any number large enough that $|x - y| < R'$ when $d(x, y) < R$, which exists by the metric bounds. In the other direction, using the same estimates for the two error terms gives an almost identical inequality

$$|\partial^k f(y) - \partial^k f(x)| \lesssim |\nabla^k f(y) - \nabla^k f(x)| + |f|_k (e^{Cd(x, y)} - 1 + C_1 |x - y|),$$

with the only difference being the constant hiding in the squiggle. Thus repeating the argument above gives the desired result. \square

We will now define a global $C^{k,\alpha}$ norm using an atlas rather than a metric, with the goal being to show that it is equivalent to the $C^{k,\alpha}$ norm arising from an arbitrary Riemannian metric. Since equivalence is transitive, this will then show that any two metrics generate equivalent norms, providing the proof of Proposition 2.4.7. Let $A' = (\mathcal{U}'_i, \varphi'_i)_{i=1}^N$ be a finite atlas for M . By [Mun00, §36 Ex 4] we can shrink this atlas to $A = (\mathcal{U}_i \subseteq \mathcal{U}'_i, \varphi_i = \varphi'_i|_{\mathcal{U}_i})$, so that each chart satisfies the hypothesis of Lemma A.2.2. Let η_i be a partition of unity subordinate to the \mathcal{U}_i and define

$$|f|_{k,\alpha;A} = \sum_i |(\eta_i f) \circ \varphi_i^{-1}|_{k,\alpha;\varphi_i(\mathcal{U}_i)}.$$

Proposition A.2.3. *For any metric g , the $C^{k,\alpha}$ norms $|\cdot|_{k,\alpha;g}$ and $|\cdot|_{k,\alpha;A}$ are equivalent.*

Proof. First note that

$$|(\eta_i f) \circ \varphi_i^{-1}|_{k,\alpha;\varphi_i(\mathcal{U}_i)} = |\eta_i f|_{k,\alpha;\delta_i}$$

where δ_i is (any extension of) the pullback metric $\varphi_i^* \delta$. When we take a j^{th} order derivative of $\eta_i f$, we will get 2^j terms of the form $\partial^I \eta_i \partial^J f$ where $|I| + |J| = j$. Since the η_i are smooth and compactly supported, this implies $|\eta_i f|_{k,0;\delta_i} \lesssim |f|_{k,0;\mathcal{U},\delta_i}$ with constant depending on η_i but not f . To handle the Hölder seminorm, write

$$\partial^k (\eta_i f) = \sum_{|I|+|J|=k} \partial^I \eta_i \partial^J f$$

and use the subadditivity of the seminorm to estimate

$$\left[\partial^k (\eta_i f) \right]_\alpha \leq \sum_{|I|+|J|=k} \left[\partial^I \eta_i \partial^J f \right]_\alpha.$$

To handle the individual terms, note that

$$\left| (\partial^I \eta_i \partial^J f)(x) - (\partial^I \eta_i \partial^J f)(y) \right| \leq \left| \partial^I \eta_i(x) \right| \left| \partial^J f(x) - \partial^J f(y) \right| + \left| \partial^J f(y) \right| \left| \partial^I \eta_i(x) - \partial^I \eta_i(y) \right|$$

and thus $\left[\partial^I \eta_i \partial^J f \right]_\alpha \leq \left| \partial^I \eta_i \right|_0 \left[\partial^J f \right]_\alpha + \left| \partial^J f \right|_0 \left[\partial^I \eta_i \right]_\alpha$. Applying this yields

$$\left[\partial^k (\eta_i f) \right]_\alpha \leq \sum_{|I|+|J|=k} \left| \partial^I \eta_i \right|_\alpha \left| \partial^J f \right|_\alpha \lesssim |f|_{k,\alpha}$$

where the constant depends only on η_i , k and α . Thus we have $|\eta_i f|_{k,\alpha;\mathcal{U},\delta_i} \lesssim |f|_{k,\alpha;\mathcal{U},\delta_i}$, which can be combined with Lemma A.2.2 to yield $|\eta_i f|_{k,\alpha;\mathcal{U},\delta_i} \lesssim |f|_{k,\alpha;g}$; so (finite) summation over i gives $|f|_{k,\alpha;A} \lesssim |f|_{k,\alpha;g}$. For the other direction, writing

$$|f|_{k,\alpha;g} = \left| \sum_i \eta_i f \right|_{k,\alpha;g} \leq \sum_i |\eta_i f|_{k,\alpha;g}$$

and applying Lemma A.2.2 gives the desired result immediately. \square

Appendix B

A Boundary Value Problem

The main results in this thesis concern closed manifolds; but it could also be very interesting to apply the same techniques to study manifolds with boundary. The simplest example is the following Cauchy-Dirichlet problem:

Problem B.0.1. Let D be the unit disc in \mathbb{R}^2 and u_0 any diffeomorphism of D fixing the boundary; i.e. $u_0|_{\partial D} = \text{id}_{\partial D}$. Is there a unique $u : [0, T) \rightarrow \text{Diff}(\bar{D})$ defined on a maximal time interval $[0, T \leq \infty)$ satisfying

$$\begin{cases} \frac{\partial u}{\partial t} = a^{ij} \nabla_i \nabla_j u & \text{in } D \times [0, T) \\ u(x, t) = x & \text{on } \partial D \times [0, T) \\ u(x, 0) = u_0(x) \end{cases} \quad (\text{B.0.1})$$

with $u(\cdot, t) \rightarrow \text{id}$ smoothly as $t \rightarrow T$? If so, is the solution map

$$\Phi : \text{Diff}(\bar{D}) \times [0, T) \rightarrow \text{Diff}(\bar{D}) : u_0 \mapsto u$$

continuous?

One strong motivation for studying this problem is to produce a new proof of the theorem of Smale [Sma59] we mentioned in the introduction: if this question is answered in the affirmative, then the flow provides a contraction of $\text{Diff}(\bar{D}, \partial D)$ to the identity.

It turns out that everything we did in Chapter 4 can also be made to work for this problem. Our maximum principle (Theorem 4.2.3) now rules out the initial bounds on the singular values being violated solely at interior points, so we just need to make sure they can't be violated at the boundary either. One standard technique for achieving boundary gradient estimates are *barriers* (as in e.g. [Lie96, Chapter X]), and it turns out they generalize quite nicely to our system. Since the Dirichlet condition already prescribes the tangential derivative of u , to bound $|Du|$ above on the boundary we just need to bound the normal derivative of u .

Let ν be the outwards unit normal to ∂D and $\rho = \frac{1}{2} (1 - |x|^2)$ a *regularized distance function* for D , by which we mean that it is a smooth function uniformly comparable to

$d_\partial : x \mapsto d(x, \partial D)$ (in this case with comparability constant 2). Note that it also agrees with d_∂ to first order at the boundary, so we can replace d_∂ with ρ in the normal derivative:

$$\nabla_\nu u(y, t) = \lim_{\epsilon \rightarrow 0} \frac{u(y, t) - u(y - \epsilon \nu_y, t)}{\epsilon} = \lim_{\substack{x \rightarrow y \\ y-x \parallel \nu_y}} \frac{u(y, t) - u(x, t)}{|y-x|} = \lim_{\substack{x \rightarrow y \\ y-x \parallel \nu_y}} \frac{u(y, t) - u(x, t)}{\rho(x)}.$$

Proposition B.0.2. *If u is a solution of (B.0.1) with a^{ij} positive-definite, then $\nabla_\nu u$ is bounded on $\partial D \times [0, T)$, with bound depending only on the initial data.*

Proof. For $y \in \partial D$ we have (remembering $u(y, t) = y$)

$$\nabla_\nu u(y, t) = \lim_{\substack{x \rightarrow y \\ y-x \parallel \nu_y}} \left(\frac{y-x}{\rho(x)} + \frac{x-u(x, t)}{\rho(x)} \right),$$

which can be written as

$$\nabla_\nu u(y, t) = y + \lim_{\substack{x \rightarrow y \\ y-x \parallel \nu_y}} q(x, t)$$

where $q : D \times [0, T) \rightarrow \mathbb{R}^2$ is defined by

$$q(x, t) = \frac{x - u(x, t)}{\rho(x)}.$$

Thus it suffices to show that q is bounded. By removing the y -dependence we have made this very easy to tackle with a barrier: for an arbitrary unit vector v and some $\Lambda > 0$ to be determined later, define the function $\varphi_v(x, t) = (x - u(x, t)) \cdot v - \Lambda \rho(x)$, so that $|q| \leq \Lambda$ if $\varphi_v \leq 0$ for all $v \in S^1$. Then since u is a solution of the flow equation, we have

$$\partial_t \varphi_v - a^{ij} \nabla_i \nabla_j \varphi_v = \Lambda a^{ij} \nabla_i \nabla_j \rho = -\Lambda \operatorname{tr} a \leq 0;$$

and by the boundary condition for u we see that φ_v vanishes on $\partial D \times [0, T)$. Thus the weak maximum principle (Theorem 2.3.9) implies that $\varphi_v \leq \sup_{t=0} \varphi_v$; so if $\varphi_v \leq 0$ at the initial time then this persists for all time. Finally, we just need to establish this at the initial time, which is not too hard: if we let $\hat{x} = x/|x| \in \partial D$ (so that $u(\hat{x}) = \hat{x}$) we can estimate

$$\varphi_v(x, 0) \leq |x - u_0(x)| - \Lambda \rho(x) \leq |x - \hat{x}| + |u_0(\hat{x}) - u_0(x)| - \frac{1}{2} \Lambda d_\partial(x) \leq (1 + |Du_0|_0 - \frac{1}{2} \Lambda) d_\partial(x),$$

so choosing $\Lambda = 2(1 + |Du_0|_0)$ to make this non-positive we get the desired uniform estimate

$$|\nabla_\nu u(y, t)| \leq 3 + 2|u_0|_1$$

for all $y \in \partial D, t \in [0, T)$. □

This estimate is fairly sloppy: using a tighter approximation to the distance function and being a little bit less eager with the triangle inequality could probably get us a better

bound. Our punishment is that the bound we get is not optimal, and it certainly isn't persistence of the equivalent bound at the initial time. This kind of sharp bound persistence would be nice to have, but is not necessary for our purposes - we just want to establish regularity, and it turns out even a time-dependent bound would be enough so long as it remained finite for all time.

We have our gradient bounds, but what about preserving diffeomorphisms? We could develop another barrier designed to give a lower bound on $\nu \cdot \nabla_\nu u$ with not too much difficulty, but in light of §3.5 and the very mild assumptions Proposition B.0.2 places on the coefficients an even easier route is open to us: we can simply apply it again, but this time to the inverse map.

Theorem B.0.3. *If a^{ij} are the coefficients of an invariant flow satisfying the hypotheses of Theorem 4.2.3 and $u : \bar{D} \times [0, T) \rightarrow \bar{D}$ is a smooth solution of $\partial_t u = a^{ij} \partial_i \partial_j u$ satisfying $u|_{\partial D} = \text{id}_{\partial D}$ and $u_0 = u|_{D \times \{0\}} \in \text{Diff}(\bar{D})$, then there are constants $0 < \lambda_- \leq \lambda_+ < \infty$ depending only on u_0 such that $\lambda_- \leq \sigma(Du) \leq \lambda_+$. In particular, $u(\cdot, t) \in \text{Diff}(\bar{D})$ for all $t \in [0, T)$.*

Proof. From the boundary condition and the fact that u_t is homotopic to u_0 (through u), we know that $u_t \in \text{Diff}(\bar{D})$ if $\sigma(Du_t) > 0$, and by continuity and the initial condition this must be true for all t in interval $[0, \epsilon)$. On this interval, both u_t and (thanks to Proposition 3.5.1) $v_t = u_t^{-1}$ are flows of diffeomorphisms satisfying the hypotheses of Proposition B.0.2, so it tells us that on $\partial D \times [0, \epsilon)$ we have $\lambda_- \leq 1/\sigma(Dv^{-1}) = \sigma(Du) \leq \lambda_+$ for some positive constants λ_-, λ_+ depending only on u_0 . By the interior gradient estimate Theorem 4.2.3, this estimate holds on all of $\bar{D} \times [0, \epsilon)$, and thus (if $\epsilon < T$) by continuity on $\bar{D} \times [0, \epsilon]$. But then (since λ_- is strictly positive) we would know that u stays a diffeomorphism for a little longer, giving the same estimate on $[0, \epsilon + \epsilon')$ for some $\epsilon' > 0$; so in fact the estimate persists for all of $[0, T)$. \square

Unfortunately, the nature of the Hölder estimate Theorem 5.3.6 makes it difficult to extend to boundary-value problems: while the barriers we cooked up above provide uniform bounds for r and the techniques of Chapter 5 would provide local Hölder estimates for θ on the interior, in order to get uniform Hölder estimates up to the boundary we would need good boundary conditions for the PDE system governing the evolution of r, θ . Unfortunately, the Dirichlet condition for u does not translate to such conditions: instead we get some kind of coupled Robin boundary conditions for r, θ , and thus the evolution of θ no longer decouples.

Appendix C

Evolution of the Rotational Component

It would be nice to have a general replacement for Proposition 5.3.5 - it is unusual that the local Hölder estimate goes through but we are held back by lack of a sup bound for θ . In order to replace it with something similar, we need some replacement for the global reasoning we used in it: when we have background curvature, θ will change by a lot more than just a constant when we change the base point of our frames. Thus the natural idea is to study Θ directly, and hope we can formulate some kind of maximum principle for it that translates in to local uniform bounds for θ . The first stepping stone towards such a result is to find an invariant evolution equation for Θ :

Proposition C.0.1. *If $u : M^2 \times [0, T) \rightarrow N^2$ is a solution of the flow (1.0.1), then its rotational component $\Theta \in \Gamma(\mathfrak{S}^* \otimes u^*TN)$ satisfies the evolution equation*

$$\nabla_t \Theta = \frac{1}{r^2} \left(\Delta \Theta + |\nabla \Theta|^2 \Theta \right). \quad (\text{C.0.1})$$

Proof. As in §5.3, we let E, V be the frames defined by parallel transport of the singular vectors at X_0 ; but this time we are just going to evaluate everything at X_0 , rather than estimating in a neighbourhood. Evaluating (5.3.9) at X_0 gives $P\theta = 0$ there; and if we let $\Psi = e^{i\theta}$ (with θ defined as in the proof of Proposition 5.3.1) and differentiate, then we quickly find that this implies $P\Psi = r^{-2} |\nabla \Psi|^2 \Psi$. Since $\Theta = V\Psi E^{-1}$ and the derivatives of V, E vanish at X_0 , we have

$$P\Theta = (PV) \Psi E^{-1} + V (P\Psi) E^{-1} + V \Psi P (E^{-1}).$$

As remarked earlier, we know that $PE = 0$ at the origin. To handle PV , recall (5.3.10), which we can write at the origin using the singular value decomposition as

$$PV_1 = -F \sum_i \sigma_i^2 (\nabla_{v_i, v_i}^2 W_1)_u.$$

Since W was constructed on N in the same manner as E on M , we know that the radial second derivatives $\nabla_{v_i, v_i}^2 W_1$ vanish at $u(X_0)$, and thus we have

$$P\Theta = r^{-2} |\nabla\Psi|^2 V\Psi E^{-1} = r^{-2} |\nabla\Psi|^2 \Theta$$

at X_0 . Since X_0 was arbitrary, we complete the proof by noting $|\nabla\Psi| = |\nabla\Theta|$, which follows from the orthonormality of the frames and the equation $\nabla\Theta = V(\nabla\Psi)E^{-1}$ at X_0 . \square

Looking at this equation, you probably have one of two reactions. My first thought was “What’s that gradient term doing there?”, but this was quickly replaced by “Hey, that’s the Laplacian of a map into a circle!”. Indeed, if we write a smooth map $\phi : M \rightarrow S^1$ in terms of the components $\phi^1, \phi^2 : M \rightarrow \mathbb{R}^2$ coming from the standard embedding $S^1 \rightarrow \mathbb{R}^2$, the map Laplacian is exactly

$$(\Delta\phi)^i = \Delta(\phi^i) + |\nabla\phi|^2 \phi^i.$$

The gradient term can be thought of as a Lagrange multiplier: it is the normal displacement necessary to keep the tension field tangent to the circle. Since the bundle $SO(\mathfrak{S}, u^*TN)$ sits inside $\mathfrak{S}^* \otimes u^*TN$ as a round circle bundle (think about the the orthogonal transformations $SO(2)$ sitting inside \mathbb{R}^4 as a round circle), equation (C.0.1) is actually very natural: it is the closest thing to the heat equation (with coefficient $1/r^2$) that Θ can satisfy while remaining orthogonal.

The expression $\tau^v(\Theta) := \Delta\Theta + |\nabla\Theta|^2 \Theta$ is known as the *vertical tension field* of the section $\Theta \in \Gamma(SO(\mathfrak{S}, u^*TN))$. One way to think of this is as the L^2 gradient of the Dirichlet energy of Θ (as a map $M \rightarrow SO(\mathfrak{S}, u^*TN)$ between manifolds with their natural Riemannian structure) on the space $\Gamma(SO(\mathfrak{S}, u^*TN))$. For more on this point of view, see the papers of C. M. Wood: these concepts were first explicitly introduced in [Woo86]. The case of unit vector fields can be found in [Woo97] and is very similar to what we have here. In [Woo90], the existence of harmonic sections (those satisfying $\tau^v(\Theta) = 0$) was shown using a heat-flow method under the assumption that the fibres have non-positive curvature; so (letting Θ_t denote the restriction of Θ to time t) the homotopy class of Θ_0 contains a harmonic section T . If the circle bundle did not have temporal curvature, then we could parallel translate this section forwards in time while preserving its harmonicity, and thus an analogous trick to that used in Proposition 5.3.5 would work to get some global control on Θ : we could write $\Theta = T + e^{i\theta}$ (notating the natural $SO(2)$ action on $SO(\mathfrak{S}, u^*TN)$ as addition) for some smooth $i\theta : M \rightarrow SO(2)$, and the equation (C.0.1) along with the fact that $\nabla_t T = \tau^v(T) = 0$ should then imply that $e^{i\theta}$ satisfies the same equation

$$\partial_t e^{i\theta} = \frac{1}{r^2} \left(\Delta e^{i\theta} + |\nabla e^{i\theta}|^2 e^{i\theta} \right),$$

which is equivalent to $\partial_t \theta = r^{-2} \Delta \theta$. Thus this *global* θ would satisfy a maximum principle,

and we could hope that the harmonicity of T is a natural enough condition that bounds on this global θ translate to uniform bounds on the local θ defined using radially parallel frames.

Unfortunately, this program is mathematical fiction for now: when N is not flat, the bundle $SO(\mathfrak{S}, u^*TN)$ has temporal curvature arising from the time-variation of u , so we have no such harmonic “gauge” T to work with. It’s quite possible that a more detailed investigation of this approach will yield a suitable T with which we can make the decomposition work, maybe in some approximate sense with controlled errors.

Instead, we will show how we can use a space-time integral estimate to derive a local supremum bound for θ . The idea is to pass from a global space-time L^2 bound for $\nabla\Theta$ to a local one for $\nabla\theta$, then to use a Poincaré-type inequality to turn this into a local L^2 bound for θ . From here the local maximum principle (Proposition 2.4.23) will provide the desired local supremum bound for θ . In order to get the L^2 bound for $\nabla\Theta$ we will still need some input from the global ideas we’ve been discussing, so while they are fresh in our minds we will prove another geometric formula for Θ .

Lemma C.0.2. *The covariant derivative of Θ can be expressed as*

$$\nabla\Theta = \frac{1}{r} \langle \nabla^2 u, J\Theta \rangle \otimes J\Theta$$

where $J \in SO(u^*TN)$ is the restriction of the complex structure of N . In particular, we can estimate $|\nabla\Theta| \leq \frac{2}{r} |\nabla^2 u|$.

Proof. As in the proof of Proposition C.0.1, we will prove this at a given point X_0 using the frames E, V , and let $\Psi = e^{i\theta} = V^{-1}\Theta E : Q(X_0, R_0) \rightarrow SO(2)$ denote the coordinate representative of Θ in these frames. Since the derivatives of E, V vanish at X_0 , at this point we have

$$V^{-1}(\nabla_j \Theta)E = \nabla_j \Psi = \partial_j \theta i\Psi.$$

Now let ξ, ζ be defined as in Proposition 5.3.1, and note that $\zeta = 0, \xi = r$, and the connection forms vanish at X_0 ; so at this point (5.3.8) along with $r^2 d\theta = \xi d\zeta - \zeta d\xi$ yields

$$\partial_j \theta = \frac{1}{r} [\nabla^2 u(\partial_j, E_1, V_2) - \nabla^2 u(\partial_j, E_2, V_1)],$$

which can be written alternatively as

$$d\theta = \frac{1}{r} \langle \nabla^2 u, V_2 \otimes E_1^b - V_1 \otimes E_2^b \rangle;$$

here we don’t have to worry about which lower index we’re contracting since $\nabla^2 u$ is symmetric. Since $\Theta = \sum_a V_a \otimes E_a^b$ at the origin, we recognize $V_2 \otimes E_1^b - V_1 \otimes E_2^b = J\Theta$; so we have

$$\nabla\Theta = \frac{1}{r} \langle \nabla^2 u, J\Theta \rangle \otimes Vi\Psi E^{-1}.$$

As $V_{X_0} : \mathbb{R}^2 \rightarrow (u^*TN)_{X_0}$ is an orientation-preserving linear isometry, we have $Vi = JV$;

so $Vi\Psi E^{-1} = J\Theta$ and we are done. \square

To interpret this geometrically, remember that Θ is a section of the circle bundle $SO(\mathfrak{S}, u^*TN)$. This fact alone constrains the derivative of Θ to lie in the span of $J\Theta$; and it turns out that (up to the factor of r) the derivative of Θ is just the orthogonal projection of the derivative of Du . To achieve the bound just notice that a 2×2 orthogonal matrix has norm $\sqrt{2}$.

Now we will derive our global space-time integral estimate:

Proposition C.0.3. *If $u : M^2 \times [0, T) \rightarrow N^2$ is a solution of a flow $\partial_t u = F\Delta u$ with a gradient estimate $\sigma(Du) \leq \lambda_+ < \infty$ and uniform ellipticity $F \geq \lambda > 0$, then for any time $T' < T$ the quantity*

$$\int_0^{T'} \int_M |\nabla^2 u| dA dt$$

is bounded by a constant depending only on geometry, λ_+ and λ - in particular, independently of T' .

Proof. Letting $E(t) = \frac{1}{2} \int |Du(t)|^2$ denote the Dirichlet energy, recall the formula

$$E'(t) = - \int F |\Delta u|^2$$

from the proof of Proposition 6.3.1. Thus we have

$$\int_0^{T'} \int F |\Delta u|^2 dA dt = - \int_0^{T'} E'(t) dt = E(0) - E(T') \leq \text{Area}(M) \lambda_+^2.$$

Estimating $F \geq \lambda$ and integrating by parts twice we get

$$\int_0^{T'} \int |\nabla^2 u|^2 dA dt \leq \text{Area}(M) \left(\frac{\lambda_+^2}{\lambda} + |R^M| + \lambda_+^2 |R^N| \right).$$

\square

Combining this with Lemma C.0.2 we get an L^2 bound for $\nabla\Theta$, and likewise for the derivatives of the coefficients $a^{ij} = F(Du)g^{ij}$:

Corollary C.0.4. *Suppose $0 < \lambda_- \leq \sigma(Du) \leq \lambda_+$ and $F \geq \lambda > 0$. For any cylinder $Q(X_0, 2R_0) \subset M \times [0, T)$, the quantities*

$$\int_{Q(X_0, 2R_0)} |\nabla\Theta|^2 \quad \text{and} \quad \int_{Q(X_0, 2R_0)} |\nabla a^{ij}|^2$$

are bounded by a constant depending only on geometry and λ_{\pm}, λ .

When $2R_0$ is small enough that we are in the setting of Lemma 5.3.2, this implies a bound on $\int_{Q(X_0, 2R_0)} |d\theta|^2$ depending only on geometry, initial data and R_0 . Since $P\theta$ can also be bounded in terms of this data, we can use this along with the Poincaré inequality

to get a local L^2 bound for (the oscillation of) θ . We will first state a weighted Poincaré inequality that we will need, which can be found with proof in [Lie96, Lemma 6.12]:

Lemma C.0.5. *Let η be a non-negative continuous function on \mathbb{R}^n with compact support Σ and $\int_{\Sigma} \eta = 1$, suppose that the sets $\{\eta \geq k\}$ are convex for $k < \sup \eta$, and set $R = \text{diam}(\Sigma)$. If $f \in W^{1,p}$, then*

$$\int_{\Sigma} |f - \bar{f}| \eta \leq C(n) (\sup \eta) R^{n+p} \int_{\Sigma} |Df|^p \eta$$

where $\bar{f} = \int_{\Sigma} \eta f$ is the η -weighted average of f .

Now to the local L^2 estimate for θ . The idea for this estimate came from [Ara16] - we have the technical convenience of not needing to work in terms of weak solutions, but the disadvantage of having to deal with an equation in general form along with a source term.

Lemma C.0.6. *Suppose that $2R_0 < \min(\iota_M, \lambda_+^{-1} \iota_N)$. Then there is a constant $\bar{\theta}_{\eta} \in \mathbb{R}$ such that $\int_{Q(X_0, R_0)} |\theta - \bar{\theta}_{\eta}|^2$ can be bounded in terms of $\int_{Q(X_0, 2R_0)} |d\theta|^2$, R_0 , geometry and singular values.*

Proof. Choose normal coordinates so that $X_0 = 0$. Let $\eta \in C_c^{\infty}(B(2R_0))$ be a monotone radial cut-off function that is equal to 1 on $B(R_0)$ and has gradient bounded by $2/R_0$. We will write

$$\theta_{\eta}(t) = \frac{\int_{B(2R_0)} \eta(x) \theta(x, t) dx}{\int_{B(2R_0)} \eta(x) dx}$$

for the η -weighted average of θ at time t , and

$$\bar{\theta}_{\eta} = \frac{1}{4R_0^2} \int_{-4R_0^2}^0 \theta_{\eta}(t) dt = \frac{\int_{Q(2R_0)} \eta(x) \theta(x, t) dx dt}{\int_{Q(2R_0)} \eta(x) dx dt}$$

for the η -weighted space-time average of θ . The result we will prove is something like a Poincaré inequality for space-time integrals, although we are crucially relaxing the estimate by requiring regularity over the enlarged cylinder $Q(2R_0)$ in order to achieve our estimate on $Q(R_0)$. Using $|\theta(x, t) - \bar{\theta}_{\eta}|^2 \lesssim |\theta(x, t) - \theta_{\eta}(t)|^2 + |\theta_{\eta}(t) - \bar{\theta}_{\eta}|^2$, we can estimate

$$\begin{aligned} \int_{Q(R_0)} |\theta - \bar{\theta}_{\eta}|^2 &\leq \int_{Q(2R_0)} \eta |\theta - \bar{\theta}_{\eta}|^2 \\ &\lesssim \int_{-4R_0^2}^0 \int_{B(2R_0)} \eta(x) |\theta(x, t) - \theta_{\eta}(t)|^2 dx dt + \int_{Q(2R_0)} \eta(x) |\theta_{\eta}(t) - \bar{\theta}_{\eta}|^2 dx dt. \end{aligned} \quad (\text{C.0.2})$$

Since small normal coordinate balls about X_0 coincide exactly with metric balls, the super-level sets of η are convex in our coordinates; so we can apply Lemma C.0.5 in coordinates with $u = \theta$, $R = 2R_0$ to get an estimate

$$\int_{B(2R_0)} \eta(x) |\theta(x, t) - \theta_{\eta}(t)|^2 dx \leq CR_0^2 \int_{B(2R_0)} \eta(x) |d\theta(x, t)|^2 dx,$$

where C depends on geometry¹. Since $\eta \leq 1$, we thus conclude that the first term in (C.0.2) is bounded in terms of $\int_{Q(2R_0)} |d\theta|^2$ and R_0 . Thus we focus our attention on the second term, starting by estimating

$$\begin{aligned} |\theta_\eta(t) - \bar{\theta}_\eta|^2 &= \left| \frac{1}{4R_0^2} \int_{-4R_0^2}^0 (\theta_\eta(t) - \theta_\eta(s)) ds \right|^2 \\ &\leq \int_{-4R_0^2}^0 |\theta_\eta(t) - \theta_\eta(s)|^2 ds \end{aligned}$$

using Hölder's inequality. Since we are allowing our bound to depend on R_0 , this means it suffices to bound $|\theta_\eta(t_1) - \theta_\eta(t_2)|$ for distinct times $t_1, t_2 \in (-4R_0^2, 0)$. Since we know from Proposition 5.3.1 and Lemma 5.3.2 that we have $P\theta = f$ with controlled $|f|_{0,Q(2R_0)}$, we can substitute $\partial_t \theta = a^{ij} \nabla_i \nabla_j \theta + f$ and integrate by parts (using $\eta \in C_c^\infty(Q(2R_0))$) to find

$$\begin{aligned} |\theta_\eta(t_2) - \theta_\eta(t_1)| &= \left| \int_{t_1}^{t_2} \int_{B(2R_0)} \eta (a^{ij} \nabla_i \nabla_j \theta + f) dx dt \right| \\ &= \left| \int_{t_1}^{t_2} \int_{B(2R_0)} (-a^{ij} \nabla_i \eta \nabla_j \theta - \eta \nabla_i a^{ij} \nabla_j \theta + f \eta) dx dt \right| \\ &\leq |a^{ij}|_0 \|\nabla \eta\|_0 \|\nabla \theta\|_1 + \|\nabla a^{ij}\|_2 \|\nabla \theta\|_2 + |f|_0 \|\eta\|_1 \end{aligned}$$

where all norms are taken over $Q(2R_0)$. Since we required $|\nabla \eta|_0 \leq 2/R_0$, our L^2 estimates for $\nabla \theta$ and ∇a^{ij} along with the uniform bound for f allow us to control this in terms of the desired data. \square

The local maximum principle (Proposition 2.4.23) now tells us that $\sup_{Q(R_0/2)} |\theta - \bar{\theta}_\eta| \lesssim \int_{Q(R_0)} |\theta - \bar{\theta}_\eta|^2 + \|f\|_{n+1;Q(R_0)}$ is bounded, so we have proved the following result and thus can remove the assumption of a supremum bound for θ from Theorem 5.3.6:

Proposition C.0.7. *If $u : M^2 \times [0, T] \rightarrow N^2$ is a solution of the flow (1.0.1) with uniform bounds $0 < \lambda_- \leq \sigma(Du) \leq \lambda_+ < \infty$ on the singular values, then the local function θ near X_0 constructed using the frames from Lemma 5.3.2 has oscillation (over the whole cylinder) bounded by a constant depending only on geometry and the initial data u_0 .*

¹Here we've conflated the Riemannian volume form and norm with their Euclidean coordinate equivalents, so we're implicitly using the equivalence between the Sobolev spaces defined using the Riemannian structure and those defined using local coordinates, which was discussed briefly in §2.4.

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Index of Notation

Section	Symbol	Meaning
	∂	Boundary of a set
	∂_i	i^{th} coordinate vector field or partial derivative
	∂_t	Time translation vector field or time derivative
§2.1	∇	Covariant derivative/Natural connection
	∇_V	... in direction V
§2.1	Δ	Laplacian or tension field
	$f_\Omega f$	Average/mean of f over Ω
§2.1	\langle, \rangle	Inner product or natural dual pairing
	$v \perp w$	Perpendicularity $\langle v, w \rangle = 0$
	v^\perp	Perpendicular hyperplane: $\{w v \perp w\}$
	$ \xi $	The Riemannian norm $\sqrt{\langle \xi, \xi \rangle}$
	\oplus	Direct/Whitney sum of vector spaces or bundles
	\otimes	Tensor product of vector spaces or bundles
	\wedge	Wedge product
	$\Lambda^k E$	k^{th} exterior power of vector space or bundle E
§2.1	\flat, \sharp	Musical isomorphisms
§A	\subseteq	Compact containment
	E^*	Dual of vector space or bundle E
	u^*	Pullback along u
§2.3.1	a, b, c	Coefficients of elliptic/parabolic equation (see P)
§3	$a^{ij}(Du)$	Coefficients of map flow
	$B(x, R)$	Ball of radius R around x
§2.1	$C(\xi)$	Contraction of (1,1)-tensor ξ
§2.4	$C^k(\cdot)/ \cdot _k$	C^k space/norm with k bounded derivatives
§2.4	$C^{k,\alpha}(\cdot)/ \cdot _{k,\alpha} / [\cdot]_{k,\alpha}$	α -Hölder space/norm/seminorm with k derivatives
§2.4	$C^{k,\alpha}(M, E)$	Sections of $E \rightarrow M$ with bounded $C^{k,\alpha}$ norm
	d	Exterior derivative of a function or differential form
	D	Differential of a map between manifolds
	\det	Determinant of a matrix or linear map
§3.4	e_i	Right-singular vector of Du
	$\text{End}(E)$	$L(E, E) = E^* \otimes E$

§3.4	$F(\sigma_k; \sigma), F_k(\sigma)$	Speed function of invariant map flow
	$F(E)$	Frame bundle of bundle E
	$F_O(E)$	Orthonormal frame bundle of E
	$GL(n, \mathbb{R}); GL(E)$	Linear automorphisms of \mathbb{R}^n or bundle E
	$\Gamma(E)$	Module of sections of E
	Γ_{jk}^i	Christoffel symbol
	id or id_Z	Identity map of object Z
	ι_M	Injectivity radius of Riemannian manifold M
§2.2	$J^k(M, N)$	Bundle of k -jets from M to N
	$J_x^k(M, N)$... with source x
	$J^k(M, N)_z$... with target z
	$J_x^k(M, N)_z$... with source x and target z
§2.4	$L^p(\cdot) / \ \cdot\ _p$	L^p -space/norm
	$L(E, F)$	Linear maps between vector spaces/bundles $E \rightarrow F$
	M, m	Domain manifold and its dimension
	N, n	Target manifold and its dimension (if different from M)
section 3.4.5, §4.1	ω_a^b	Connection form $\langle \nabla e_a, e_b \rangle$
	$\Omega^p(M)$	$\Gamma(\Lambda^p TM^*)$
	$\Omega^p(M; E)$	$\Gamma(\Lambda^p TM^* \otimes E)$
	$O(V), O(n)$	Orthogonal group of inner product space V or \mathbb{R}^n
§2.3.1	P	Parabolic operator $\partial_t - a^{ij} \nabla_i \nabla_j$
§6	P_u	Frozen operator $\partial_t - a^{ij} (Du) \nabla_i \nabla_j$
§2.3.1	$\mathcal{P}\Omega$	Parabolic boundary of Ω
§2.3.1	$Q(X, R)$	Parabolic neighbourhood of “radius” R “around” X
§3.1	\mathfrak{S}	Spatial tangent bundle $TM \times \mathbb{R} \subset T(M \times \mathbb{R})$
§3.4.2	$SO(E, F)$	Bundle of oriented linear isometries $E \rightarrow F$
	$\text{Sym}^k E$	k^{th} symmetric power of vector space or bundle E
§3.4	σ_i	Singular value of Du
section 3.4.5, §4.1	τ_a^b	Connection form $\langle \nabla v_a, v_b \rangle$
	TM	Tangent bundle of manifold M
	tr	Trace of a matrix or (1,1)-tensor
§2.1	tr_g	Trace of a tensor with respect to metric g
§3–§6	u	(maybe time-dependent) map between manifolds
§3.4	v_i	Left-singular vector of Du
§2.4	$W^{k,p}(\cdot) / \ \cdot\ _{W^{k,p}}$	Sobolev space/norm with k derivatives, exponent p
§2.3.1	X	A point (x, t) in space-time $M \times \mathbb{R}$.

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